# MATH 231A: LECTURE 11 REFERENCES

## 1. Non-Linear Variational Problems and the Regularity Problem

- The subject of non-linear problems arising from variational problems is vast and well beyond my ability to give even partially complete references. Chapter 8 of Evans is a good place for the interested reader to start. After that, I would recommend reading Giaquinta's notes [2] which are devoted to regularity problems, and which contains a nice discussion of many estimates (including Schauder) for elliptic PDE. See also Chapter 12.3 of Jost's book (on reserve in the library), which covers the special case of uniformly elliptic energy densities of the form $F(\nabla u)$ with quadratic growth.

  Roughly speaking, the real work in understanding the regularity of solutions to variational problems is to establish some basic threshold of smoothness for the solution. This is in the form of a $C^\alpha$ bound for either the solution or its gradient. With this threshold established, then it is "classical" to upgrade to full regularity (i.e. by repeatedly using Schauder estimates as we have discussed). It is important to note that for *any* solution to the corresponding elliptic PDE, assuming a certain threshold regularity (basically a $C^0$ estimate for the solution or its gradient, depending on the context) implies complete regularity of the solution. This is just a generic feature of non-linear elliptic equations, and has nothing to do with variational problems.

  What is most interesting is that there are natural variational problem where the basic "starter regularity" *cannot* be established. This is more than a technical problem, these variational problems have discontinuous solutions, even when restricting to the class of energy minimizers! (See for example the HW problem on harmonic maps). Roughly speaking, this is a problem that can only effect systems, and there is a more or less complete regularity theory for minimizers to scalar variational problems.

- For further reading on variational problems, especially ones arising in geometry and physics, I would recommend looking into L. Simon's notes [6], and then Giaquinta's classic [1] or the book [4] for a thorough and complete coverage of a wide class of problems.

- For more on regularity of non-linear problems, especially the relationship between proving a-priori estimates and establishing the existence of classical solutions to non-linear problems, see Gilbarg-Trudinger [3]. There is a discussion of similar material in the later chapters of [7].

## References


[1] Giaquinta, Mariano **Multiple integrals in the calculus of variations and nonlinear elliptic systems.** Annals of Mathematics Studies, 105. Princeton University Press, Princeton, NJ, 1983.

[2] Giaquinta, Mariano **Introduction to regularity theory for nonlinear elliptic systems.** Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993. viii+131 pp.

[3] Gilbarg, David; Trudinger, Neil S. **Elliptic partial differential equations of second order.** Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp.

[4] Giusti, Enrico **Direct methods in the calculus of variations.** World Scientific Publishing Co., Inc., River Edge, NJ, 2003. viii+403 pp.

[5] Han, Qing; Lin, Fanghua **Elliptic partial differential equations.** Courant Lecture Notes in Mathematics, 1. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997. x+144 pp.

[6] Simon, Leon **Theorems on regularity and singularity of energy minimizing maps. Based on lecture notes by Norbert Hungerbühler.** Lectures in Mathematics ETH Zrich. Birkhuser Verlag, Basel, 1996. viii+152 pp.

[7] Wu, Zhuoqun; Yin, Jingxue; Wang, Chunpeng **Elliptic & parabolic equations.** World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006. xvi+408 pp.