

MATH 231A: 7 LECTURE REFERENCES

1. INTERIOR SCHAUDER ESTIMATES

- Schauder estimates for the Laplacian are classical, and are treated in many different texts. There are at least three standard approaches to proving Schauder estimates:

- (1) Energy estimates and Campanato-Morrey spaces.
- (2) Direct estimation of convolutions with fundamental solutions.
- (3) L. Simon's rescaling argument.

We briefly discuss each method and associated references below. For some history and additional reference refer Chapter 6 of Gilbarg and Trudinger [2], which is a classical text on both linear and non-linear elliptic equations.

- The Campanato lemma is standard, and proved in Section 3.2 of [3], Section 3.1 of [1], and Section 6.1 of [5]. The idea behind the estimate is very similar to Morrey's inequality, proved in Section 5.6.2 of Evans.

2. VARIOUS PROOFS OF SCHAUDER ESTIMATES

2.1. Method based on energy estimates. This is the technique we follow in the notes and in class. A basic treatment is given in Chapter 3 of [3], Chapter 3 of [1], and Chapter 6 of [5].

The advantages of this approach is that it is east to modify to deal with C^0 coefficients a_{ij} and naturally gives sharp L^p indices for the RHS. Also, the treatment of boundaries is also straight forward once one can prove the associated decay of integrals for harmonic functions.

2.2. Method based on the fundamental solution. This is the other "standard" approach to Schauder estimates. The idea is to write the solution to equation $\Delta w = F$ in convolution form:

$$w(x) = c_n \int_{\mathbb{R}^n} \frac{1}{|x-y|^{n-2}} F(y) dy ,$$

and then to carefully estimate the differences $|w(x_1) - w(x_0)|$ by studying the corresponding integrals. This is the approach taken in Sections 4.3 and 4.5 of Gilbarg-Trudinger [2], and also in Chapter 11 of J. Jost's book (on reserve in the library).

While this approach leads directly to the classical $C^{2,\alpha}$ Schauder estimates, a bit more work is required to deal with L^p space RHS and it is not as natural when extending estimates to a_{ij} with C^0 coefficients.

2.3. Method based on rescaling. This is probably the most efficient proof of Schauder estimates. An exposition is given in Section 1.7 of L. Simon's beautiful notes on harmonic mappings [4]. The idea here is very similar to how one proves inequalities like Poincaré's: one assumes there is a sequence of functions $w^{(n)}$ which break the estimate, and then uses some kind of compactness argument to extract a subsequence converging to a function provides an obviously false situation.

In this case, we choose our sequence of test functions such that (\dot{C}^α refers to just the difference quotient portion of C^α , which is homogeneous):

$$\|\nabla^2 w^{(n)}\|_{\dot{C}^\alpha(\mathbb{R}^n)} \geq n \|\Delta w^{(n)}\|_{\dot{C}^\alpha(\mathbb{R}^n)}.$$

By rescaling both sides of this inequality, and then passing to a limit, one obtains a simple contradiction involving polynomial functions (basically one ends up with limit that must be a non-linear 2nd degree polynomial such that $\nabla^2 w^{(\infty)}(0) = 0$, which is impossible).

REFERENCES

- [1] Giaquinta, Mariano **Introduction to regularity theory for nonlinear elliptic systems.** Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1993. viii+131 pp.
- [2] Gilbarg, David; Trudinger, Neil S. **Elliptic partial differential equations of second order.** Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001. xiv+517 pp.
- [3] Han, Qing; Lin, Fanghua **Elliptic partial differential equations.** Courant Lecture Notes in Mathematics, 1. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997. x+144 pp.
- [4] Simon, Leon **Theorems on regularity and singularity of energy minimizing maps. Based on lecture notes by Norbert Hungerbühler.** Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel, 1996. viii+152 pp.
- [5] Wu, Zhuoqun; Yin, Jingxue; Wang, Chunpeng **Elliptic & parabolic equations.** World Scientific Publishing Co. Pte. Ltd., Hackensack, NJ, 2006. xvi+408 pp.