SCHAUDER THEORY II: BOUNDARY ESTIMATES FOR LAPLACE

JACOB STERBENZ

Abstract. These are companion lecture notes for Math 231A on Friday 10/16/09.

1. Boundary Schauder Estimates

The purpose of these notes is to prove estimates of Schauder type for solutions to the equation $\Delta w = F$ assuming that the domain of definition has a simple boundary. We'll build upon these estimates later to prove a much more general and powerful set of Schauder estimates that are strong enough to provide existence of classical solutions for general second order elliptic operators.

The setup for the estimates here is the following: Recall that previously we have shown estimates which are schematically of the form:

$$\| w \|_{Holder(\Omega')} \leqslant C(\| \Delta w \|_{N(\Omega)} + \| w \|_{weaker(\Omega)}) , \tag{1}$$

where $\Omega' \subset\subset \Omega$ is a strict inclusion, and the constant C blows up like a power of $dist(\Omega, \Omega')$, and where N is a norm of "two derivatives less" than the Hölder norm on the LHS, and "weaker" denotes a much weaker norm of w (in fact H^1). We would like to extend (1) to the situation where Ω' and Ω share a common boundary, at least along some of $\partial\Omega'$ and $\partial\Omega$.

To express the situations we will handle here, we employ the following notation. If $\Omega \subset \mathbb{R}^n$ is any subdomain, we set:

$$\Omega^+ \ = \ \{(x', x^n) \in \mathbb{R}^n \,\big|\, x^n > 0\} ,$$

and Ω^- with a similar definition involving $x^n < 0$. Our intention is to prove the following:

Theorem 1.1 (Half-Space Schauder Estimates)**.** *Let $w \in H^1(\Omega^+)$ be a weak solution to*

$$\iint_{\Omega^+} \nabla w \cdot \nabla \varphi \ dx \ + \iint_{\Omega^+} F \varphi \ dx \ = \ 0 , \qquad \forall \ \varphi \in C_0^\infty(\Omega^+) , \tag{2}$$

with the additional property that $T_{\Omega^+} w|_{x^n=0} = 0$. Then for every $\alpha \in (0,1)$ there exists a uniform constant $C = C(\alpha, \Omega)$ such that if $\Omega' \subset\subset \Omega$ is any smooth subdomain with $d = d(\Omega, \Omega') = dist(\partial\Omega, \partial\Omega')$ then the following estimates hold:

- *i) If $F = f + \sum_i \partial_i f_i$ (weakly) where $f \in L^p(\Omega^+)$ and $f_i \in L^q(\Omega^+)$ for $\alpha = 2 - \frac{n}{p} = 1 - \frac{n}{q} \in (0,1)$, then $w \in C^\alpha(\Omega^+)$ and:*

$$\| w \|_{C^\alpha(\Omega'^+)} \leqslant Cd^{-\frac{n}{2}+1-\alpha} \left(\| f \|_{L^p(\Omega^+)} + \| (f_i) \|_{L^q(\Omega^+)} + \| w \|_{H^1(\Omega^+)}\right) . \tag{3}$$

ii) If instead assume that $f \in L^p(\Omega^+)$ for $\alpha = 1 - \frac{n}{p} \in (0,1)$, and $f_i \in C^\alpha(\Omega^+)$, then $\nabla w \in C^\alpha(\Omega^+)$ and:

$$\| \nabla w \|_{C^\alpha(\Omega'^+)} \leqslant C d^{-\frac{n}{2}-\alpha} \big(\| f \|_{L^p(\Omega^+)} + \| (f_i) \|_{C^\alpha(\Omega^+)} + \| w \|_{H^1(\Omega^+)} \big) \ . \tag{4}$$

Proof. The proof of the estimates (3)–(4) follows *exactly* the same pattern as the case without the $x^n = 0$ boundary component (possibly) shared by Ω' and Ω. We only need to establish three things:

1) **Integral decay estimates in the case where $F \equiv 0$.** That is, bounds for harmonic functions. This is very similar to the interior case done previously, with one small twist.
2) **A harmonic approximation lemma.** There is no difference between this and the interior case, as the approximation only depends on existence for the Dirichlet problem in H^1 and the triangle inequality.
3) **Integral decay estimates for solutions of $\Delta w = F$.** This also uses the exact same calculations as in the interior case: just integration by parts, Hölder's inequality, and the iteration lemma.

In the next section we deal with 1) above, while steps 2)–3) are left to the reader (with the statement of the estimates for step 1 the later steps should be obvious, but the reader is encouraged to fill out the steps of the proof to be sure). □

2. Integral Decay for harmonic Functions on Half-Spaces

Here we fill out the details of step 1) from the previous section. We'll prove the following:

Lemma 2.1 (Local Decay of Integrals for Harmonic Functions on Half-Spaces)**.** *Let $\Delta u = 0$ be in $H^1(\Omega^+)$ be such that $T_{\Omega^+} u|_{x^n=0} = 0$. Then there exists a universal constant $C > 0$ such that for any pair of concentric balls $B^+_\rho(x_0) \subseteq B^+_r(x_0) \subseteq \Omega^+$ one has:*

$$\iint_{B^+_\rho(x_0)} |\nabla u|^2 dx \ \leqslant\ C \left(\frac{\rho}{r}\right)^n \iint_{B^+_r(x_0)} |\nabla u|^2 dx \ , \tag{5}$$

$$\iint_{B^+_\rho(x_0)} |\nabla u - \underline{\nabla u}_{x_0,\rho}|^2 dx \ \leqslant\ C \left(\frac{\rho}{r}\right)^{n+2} \iint_{B_r(x^+_0)} |\nabla u - \underline{\nabla u}_{x_0,r}|^2 dx \ , \tag{6}$$

where $\underline{f}_{x_0,r} = |B^+_r(x_0)|^{-1} \int_{B^+_r(x_0)} f \, dx$.

The proof of this Lemma relies on a simple result which the reader may be familiar with in the context of holomorphic functions:

Proposition 2.2 (The Schwartz Reflection Principle)**.** *Let $w \in H^1(\Omega^+)$ be weakly harmonic in Ω^+ and such that $T_{\Omega^+} u|_{x^n=0} = 0$. Then if we define the reflection Ω^- of Ω^+ according to the rule:*

$$\Omega^- \ = \ \{(x', x^n) \in \mathbb{R}^n \,|\, (x', -x^n) \in \Omega^+\} \ ,$$

and $\Omega = int\big(\Omega^+ \cup \Omega^- \cup \{x^n = 0\}\big)$ to be their union including all points $x^n = 0$ in the intersection of there two boundaries, then if we also define $\widetilde{u}$ according to the formula:

$$\widetilde{u}(x', x^n) \ = \ -u(x', -x^n) \ , \qquad \textit{for} \ \ (x', x^n) \in \Omega^- \ ,$$

we have that $\widetilde{u} \in H^1(\Omega)$ and also that $\widetilde{u}$ is weakly harmonic in Ω. In particular, $\widetilde{u}$ is a real analytic function on Ω which vanishes along $\{x^n = 0\} \cap \Omega$.

Remark 2.3. Notice that the assumption that $u \in H^1(\Omega^+)$ is a crucial component of the above proposition. For example, the function:

$$u(x,y) \ = \ \frac{y}{x^2+y^2} ,$$

is easily seen to be harmonic in $(\mathbb{R}^2)^+$, and in some sense we must have that $u(x,0)=0$ because its boundary limits are zero almost everywhere. But its immediate that there is no harmonic function $\widetilde{u}$ on all of $\mathbb{R}^2$ which extends u.

Proof. We need to show that:

$$\iint_{\Omega^+\cup\Omega^-} \nabla\widetilde{u}\cdot\nabla\varphi \ dx \ = \ 0 \ , \tag{7}$$

for all $\varphi\in C_0^\infty(\Omega)$. For any such function we may split it smoothly into two pieces $\varphi=\varphi_0+\varphi_1$ where $\varphi_0(x',x^n)=\chi(x^n)\varphi(x',0)$ with $\chi(x^n)$ a smooth even function, and where $\varphi_1\in H_0^1(\Omega^+)$ as well as $\varphi_1\in H_0^1(\Omega^-)$. For the second term in the splitting, we have the separate bounds:

$$\iint_{\Omega^+}\nabla u\cdot\nabla\varphi_1 \ dx \ = \ 0 \ , \qquad \iint_{\Omega^-}\nabla\widetilde{u}\cdot\nabla\varphi_1 \ dx \ = \ 0 \ ,$$

where the first identity follows from the definition of u, while the second follows from making the change of variable $y^n=-x^n$ which converts Ω^- to Ω^+ and swaps $\nabla\widetilde{u}$ with ∇u.

For the term involving φ_0 in (7) we use the antisymmetry of $\widetilde{u}$ and Fubini's theorem to integrate out the x^n variable first which yields (a.e. in x' where the integral is a-priori finite):

$$\int_{\mathbb{R}^-}\nabla\big(-u(x',-x^n)\big)\cdot\nabla\big(\chi(x^n)\varphi(x',0)\big)dx^n = -\int_{\mathbb{R}^+}\nabla\big(u(x',x^n)\big)\cdot\nabla\big(\chi(x^n)\varphi(x',0)\big)dx^n \ .$$

This identity follows from examining separately the $\partial_{x'}=\nabla'$ and ∂_{x^n} components of the gradient, and using the fact that $\chi(x^n)$ is even. □

Now armed with Proposition 2.2 we commence with the proof of (5)–(6). The proof of the first estimate is immediate from the $L^1(\Omega)\to C^k(\Omega')$ estimate for harmonic functions proved in the previous notes. The reader is left to iron out the details. Notice that we only need to make use of the fact that $\|\widetilde{u}\|_{L^1(\Omega^+\cup\Omega^-)} = 2\|u\|_{L^1(\Omega^+)}$ where Ω^- and $\widetilde{u}$ are defined as in the previous proposition.

It remains to prove (6), which involves a slight twist because one needs to handle the averages $\underline{u}_{x_0,\rho}$ properly when the ball $B_\rho(x_0)$ cuts the boundary $x^n=0$. From the usual $L^1(\Omega)\to C^k(\Omega')$ estimate and Poincare's inequality applied to the extension $\widetilde{u}$ we have the bound (rescaling to $r=1$ as usual):

$$\iint_{B_\rho^+(x_0)}|\nabla u-\underline{\nabla u}_{x_0,\rho}|^2dx \ \leqslant C\rho^{n+2}\iint_{B_1^+(x_0)}|\nabla\widetilde{u}-\underline{\nabla u}_{x_0,1}|^2dx \ .$$

(Notice that the constant $\underline{\nabla u}_{x_0,\rho}$ can be changed to anything on the RHS of Poincare because the entire expression is differentiated.) The main issue with the last estimate is that the term $|\nabla'\widetilde{u}-\underline{\nabla' u}_{x_0,1}|$ switches to $|\nabla' u+\underline{\nabla' u}_{x_0,1}|$ under the change of variables $y^n=-x^n$, so one needs to show a uniform bound of the form:

$$\iint_{B_1^+(x_0)}|\nabla' u+\underline{\nabla' u}_{x_0,1}|^2dx \ \leqslant C\iint_{B_1^+(x_0)}|\nabla' u-\underline{\nabla' u}_{x_0,1}|^2dx \ ,$$

for all $H^1(B_1^+(x_0))$ harmonic functions u with the additional property that the restriction $T_{B_1^+(x_0)}u|_{x^n=0} = 0$. In fact, we have the much more general bound:

$$\iint_{B_1^+(x_0)} |u|^2 dx \ \leqslant \ C \iint_{B_1^+(x_0)} |u-\lambda|^2 dx \ , \tag{8}$$

for a uniform constant C, *independent* of λ if we in addition assume that $u \in H^1(B_1^+(x_0))$ is such that $u|_{x^n=0} = 0$ provided $|B_1^+(x_0) \cap \{x^n = 0\}|_{\mathcal{L}^{n-1}} \geqslant c$ (this latter assumption is of course what separates the bounds (5)–(6) from their interior versions). To see this, notice that for any $L^2(B_1^+(x_0))$ function we have by the reverse triangle inequality the estimate:

$$\big| \, \| \, u \, \|_{L^2(B_1^+(x_0))} - |\lambda| \, \big| \ \leqslant \ \| \, u - \lambda \, \|_{L^2(B_1^+(x_0))} \ .$$

Thus, we may assume that we are in the range:

$$\frac{1}{2} \| \, u \, \|_{L^2(B_1^+(x_0))} \ \leqslant \ |\lambda| \ \leqslant \ 2 \| \, u \, \|_{L^2(B_1^+(x_0))} \ ,$$

for otherwise estimate (8) is immediate. Without loss of generality, assume that $\lambda = 1$ and $\| \, u \, \|_{L^2(B_1^+(x_0))} \leqslant 2$. We need a universal positive lower bound for the quantity $\| \, u - 1 \, \|_{L^2(B_1^+(x_0))}$. Doubling u to $\widetilde{u}$ as in Proposition 2.2, we have from the usual estimates for harmonic functions that:

$$\sup_{B_r(x_0)^+} |\nabla u| \ \leqslant \ C \ ,$$

where $r < 1$ and C depends on r but not u (this uses the normalization of $\| \, u \, \|_{L^2(B_1^+(x_0))}$). Integrating this last line from $x^n = 0$, we have in a small ϵ-neighborhood $\mathcal{N}_\epsilon$ of $\{x^n = 0\} \cap B_r^+$ the estimate $|u| \leqslant \epsilon C$. Integrating this over $\mathcal{N}_\epsilon^+$ we then have:

$$\frac{1}{2} \epsilon^{\frac{n}{2}} \ \leqslant \ \| \, u - 1 \, \|_{L^2(\mathcal{N}_\epsilon^+)} \ \leqslant \ \| \, u - 1 \, \|_{L^2(B_1^+(x_0))} \ ,$$

assuming $\epsilon \ll 1$, as desired. This concludes our demonstration of (5) and (6).

Department of Mathematics, University of California, San Diego (UCSD) 9500 Gilman Drive, Dept 0112 La Jolla, CA 92093-0112 USA

E-mail address: jsterben@math.ucsd.edu