SCHAUDER THEORY IV: APPLICATIONS

JACOB STERBENZ

Abstract. These are companion lecture notes for Math 231A on Wednesday 10/21/09 and Friday 10/23/09.

1. Existence of Classical Solutions to General Elliptic Equations

In this section we study the solvability properties of general equations of the form:

$$Lw \ = \ \sum_{i,j} a_{ij}(x)\partial_i\partial_j w + \sum_i b_i(x)\partial_i w + cw \ = \ F \ , \qquad \text{in} \ \ \Omega \ , \tag{1a}$$

$$u\big|_{\partial\Omega} \ = \ g \ . \tag{1b}$$

We will make the uniform ellipticity condition throughout Ω:

$$\sum_{i,j} a_{ij}\xi_i\xi_j \ \geqslant \ c|\xi|^2 \ , \qquad (x,\xi) \in \Omega \times \mathbb{R}^n \ , \tag{2}$$

and we also assume that regularity condition:

$$a_{ij} \ \in \ C^\alpha(\overline{\Omega}) \ , \quad b_i \ \in \ C^\alpha(\overline{\Omega}) \ , \quad c \ \in \ C^\alpha(\overline{\Omega}) \ . \tag{3}$$

With these assumptions, our main result is the following:

Theorem 1.1 (Existence of Classical Solutions)**.** *Let a, b, c be given so that both* (1) *and* (3) *hold. Then if $c(x) \leqslant 0$ throughout Ω, for each $F \in C^\alpha(\overline{\Omega})$ and $g \in C^{2,\alpha}(\partial\Omega)$ there exists a unique solution $w \in C^{2,\alpha}(\overline{\Omega})$ to the problem* (2)*. Furthermore, one has the uniform estimate:*

$$\| \, w \, \|_{C^{2,\alpha}(\overline{\Omega})} \ \leqslant \ C(\| \, F \, \|_{C^\alpha(\overline{\Omega})} + \| \, g \, \|_{C^{2,\alpha}(\partial\Omega)}) \ , \tag{4}$$

where $C = C(\Omega, \alpha, \| \, a \, \|_{C^\alpha(\overline{\Omega})}, \| \, b \, \|_{C^\alpha(\overline{\Omega})} \| \, c \, \|_{C^\alpha(\overline{\Omega})})$ is independent of the data (F, g).

Remark 1.2. The condition $c(x) \leqslant 0$ in the preceding theorem is very important and cannot be dropped. For example, on any domain Ω there exists (non-trivial) solutions to the Dirichlet eigenvalue problem:

$$\Delta u \ = \ -\lambda u \ , \qquad u\big|_{\partial\Omega} \ = \ 0 \ , \tag{5}$$

for certain discrete $\lambda > 0$. Furthermore, for certain highly symmetric domains like balls the $|x| \leqslant R_0$, the solutions to (5) can have large multiplicity (the eigenspaces are always finite dimensional, but they can grow in multiplicity like $O(\lambda^{\frac{n-2}{2}})$). Finally, even for simple ODE of the form:

$$\frac{d^2}{dx} u + k^2 u \ = \ F \ , \qquad \text{in} \ \ [0, \pi] \ , \tag{6}$$

there could be no solution with (say) $u(0) = u(\pi) = 0$. For example, choosing $k = 1$ and $F(x) = \sin(x)$ one can see there is no solution u to (6) vanishing at the endpoints of $[0,\pi]$, for if this was so, then integration of both sides of (6) against $\sin(x)$ yields a contradiction (this reasoning applies to the equation $(L-\lambda)u = F$ in general for eigenvalues λ of L).

To reach a proof of Theorem 1.1 we will use the Schauder estimates and two additional pieces of information. The first is interesting in its own right as it is a central a-priori estimate for second order elliptic equations with many important generalizations:

Theorem 1.3 (Weak Maximum Principle)**.** *Let $w \in C^2(\overline{\Omega})$ be a solution to the problem (2)–(3) where in addition we assume that $c \leqslant 0$. Then there exists a constant $C = C(\Omega, a, \| b \|_{L^\infty(\Omega)})$ such that the following a-priori estimate holds:*

$$\sup_{x\in\Omega} |u(x)| \ \leqslant\ C(\sup_{x\in\Omega} |F(x)| + \sup_{x\in\partial\Omega} |g(x)|) \ . \tag{7}$$

Next, we need a simple continuity lemma whose proof basically boils down the contraction mapping principle:

Lemma 1.4 (Continuity Lemma)**.** *Let A, B be Banach spaces, and suppose that $L, L_0 : A \to B$ are continuous linear operators with the additional property that the following a-priori bound holds:*

$$\| w \|_A \ \leqslant\ C \| tL + (1-t)L_0 w \|_B \ , \qquad t \in [0,1] \ . \tag{8}$$

Then if for each $F \in B$ there exists a solution $w \in A$ to $L_0 w = F$, the same property holds for the equation $Lw = F$.

Before proving these two additional results, let us conclude with a proof of Theorem 1.1. Without loss of generality we may assume that $g \equiv 0$ for otherwise we may subtract off an extension $\widetilde{g}$ and then set $\widetilde{w} = w - \widetilde{g}$ which solves (1) with $\widetilde{F} = F - L\widetilde{g}$ with a C^α estimate for $\widetilde{F}$ in terms of the appropriate norms on F and g.

Using the maximum principle bound (7) for the operator $L_t = tL + (1-t)\Delta$ and the Schauder estimates for uniformly elliptic equations with C^α coefficients we have the a-priori bound:

$$\| w \|_{C^{2\alpha}(\overline{\Omega})} \ \leqslant\ C \| L_t w \|_{C^\alpha(\overline{\Omega})} \ .$$

Furthermore, using the direct methods discussed earlier, as well as the Schauder estimates, we know that the problem:

$$L_0 w \ = \ \Delta w \ = \ F \ , \quad \text{in } \ \Omega \ , \qquad u\big|_{\partial\Omega} \ = \ 0 \ ,$$

has a solution $w \in C^{2,\alpha}(\overline{\Omega})$. Thus, we may apply Lemma 1.4 to conclude that the equation (2) has a $C^{2,\alpha}$ solution with the bound (4) as well.

1.1. **Proof of the supporting propositions.** Here we deal with Theorem 1.3 and Lemma 1.4.

Proof of Theorem 1.3. Without loss of generality, we'll assume that $\sup_\Omega |w| = \sup_{\Omega'} w > 0$, for otherwise we could consider $-w$ (or $w = 0$). Our goal is to show the estimate:

$$\sup_{\Omega} w \ \ \leqslant\ C \sup_{x\in\Omega} |F(x)| + \sup_{x\in\partial\Omega} |g(x)|) \ . \tag{9}$$

We make a further quick reduction that we may restrict (9) to the subdomain $\Omega \supseteq \Omega' = \{x \in \Omega | w(x) > 0\}$, for it is trivial otherwise. Notice that the restriction $w|_{\partial\Omega'}$ is only nonzero along the common intersection $\partial\Omega' \cap \partial\Omega$, so the assumption $\sup_\Omega w > 0$ eliminates any portion of $\partial\Omega' \setminus \partial\Omega$ from the RHS of the restricted version of (9). To recap, we are now trying to show (9) for solutions w to (1a) in domains Ω' where w is *non-negative*.

We now consider the family of functions:

$$w^+_{\epsilon,M} \ = \ w + e^{M|x-x_0|^2} \cdot (\epsilon + \sup_{x\in\Omega} |F(x)|) \ ,$$

for some $x_0 \notin \overline{\Omega'}$. A direct calculation shows that for any $0 < \epsilon \leqslant 1$, and if $M \gg 1$ is sufficiently large, (notice that this only depends on a and $\| b \|_{L^\infty(\Omega')}$) then we have the pointwise inequality:

$$\sum_{i,j} a_{ij}\partial_i\partial_j w^+_{\epsilon,M}(x) + \sum_i b_i \partial_i w^+_{\epsilon,M}(x) \ \geqslant \ \epsilon \ . \tag{10}$$

Now, let $x_0 \in \Omega'$ be a supposed interior critical point of $w^+_{\epsilon,M}$ corresponding to a local max. Then we have $\nabla w^+_{\epsilon,M}(x_0) = 0$ and also that $\nabla^2 w^+_{\epsilon,M}(x_0)$ is negative definite. But this contradicts (10) in light of (2). Thus, $w^+_{\epsilon,M}(x_0)$ has no interior maximum and we therefore have the global bound:

$$\sup_{x\in\Omega'} w^+_{\epsilon,M}(x) \ \leqslant \ \sup_{x\in\partial\Omega'} w^+_{\epsilon,M} \ .$$

Taking $\epsilon \to 0$ in this last estimate we have:

$$\sup_{\Omega'} w \ \leqslant \ \sup_{\Omega'} w^+_{0,M} \ \leqslant \ \sup_{\partial\Omega'} w + C(M) \sup_\Omega |F| \ ,$$

as was to be shown. □

Proof of Lemma 1.4. Suppose that we can always solve $L_{t_0} w = F$ for some $t_0 \in [0,1]$. Then the equation $L_t w = F$ can be written in the perturbative form:

$$L_{t_0} w \ = \ (t_0 - t) L w + (t - t_0) L_0 w + F \ , \tag{11}$$

and our task is to find a solution to this last equation. For this equation we set up a Picard iteration scheme:

$$L_{t_0} w^{(k)} \ = \ (t_0 - t) L w^{(k-1)} + (t - t_0) L_0 w^{(k-1)} + F \ ,$$

where $w^{(0)}$ solves $Lw_{t_0} w^{(0)} = F$. Then using the continuity of L, L_0 and the solvability bound (8) we have the inductive estimates:

$$\| w^{(1)} - w^{(0)} \|_A \ \leqslant \ C|t - t_0| \cdot \| w^{(0)} \|_A \ ,$$
$$\| w^{(k)} - w^{(k-1)} \|_A \ \leqslant \ C|t - t_0| \cdot \| w^{(k-1)} - w^{(k-2)} \|_A \ ,$$

for a fixed constant $C > 0$. Then choosing $|t - t_0| \ll 1$ sufficiently small we have that the $w^{(k)}$ form a Cauchy sequence which converges to a solution of (11). □

2. Regularity of Critical Points of variational Problems

In this section we consider another typical application of Schauder estimates, which is to provide full regularity of solutions to non-linear problems once one has some crucial amount of "starter regularity" to begin with.

A classical problem is to understand the regularity properties of solutions minimization problems of the form:

$$I[u] \ = \ \inf_{\substack{w\in H^1(\Omega)\\ T_\Omega u = g}} I[w] \ ,$$

where $T_\Omega u = g$ and I is of the form:

$$I[u] \ = \ \iint_\Omega F(u,\nabla u)\ dx \ . \tag{12}$$

Here $F(u,\eta) : \mathbb{R}^{n+1}\to\mathbb{R}$, $F\in C^\infty$, is some "energy density". Typical examples are:

$$F \ = \ \frac{1}{2}|\eta|^2 \ , \qquad F \ = \ \sqrt{1+|\eta|^2} \ , \qquad F \ = \ \frac{1}{2}\sum_{i,j} A_{ij}(u)\eta_i\eta_j \ , \tag{13}$$

which represent the usual (Laplace) Dirichlet energy, the minimal surface density, and a general non-linear version of the Dirichlet density (assuming that A_{ij} is uniformly elliptic).

A calculation involving variations of the form $u_t = u + t\varphi$ shows that minimizers to (12) solve the weak equation:

$$\iint_\Omega \big(F_\eta(u,\nabla u)\cdot\nabla\varphi + F_u(u,\nabla u)\varphi\big)\ dx \ = \ 0 \ , \qquad \forall \ \ \varphi\in C_0^\infty(\Omega) \ . \tag{14}$$

In other words, u weakly solves the quasilinear divergence form equation:

$$\sum_{i,j}\partial_i\big[A_{\eta^i}(u,\nabla u)\big] \ = \ F_u(u,\nabla u) \ . \tag{15}$$

Under certain structural assumptions, (15) will be a uniformly elliptic equation. For example, in the case of the last energy density on line (13), the equation reads:

$$\sum_{i,j}\partial_i\big[A_{ij}(u)\partial_j u\big] \ = \ \frac{1}{2}\sum_{i,j} A'_{ij}(u)\partial_i u\partial_j u \ . \tag{16}$$

For this equation, we have the following simple applications of the Schauder estimates we have proved thus far:

Proposition 2.1. *Let $u\in H^1(\Omega)$ be a weak solution to* (16)*, with $A_{ij}(u)$ a smooth function of its argument and such that:*

$$\sum_{i,j} A_{ij}(u)\xi_i\xi_j \ \geqslant \ c(u)|\xi|^2 \ , \qquad c>0 \ . \tag{17}$$

Assume also that u has $C^\infty(\partial\Omega)$ boundary data g. If we assume *that $\nabla u\in L^\infty(\Omega)$ then we have $u\in C^\infty(\overline{\Omega})$.*

Remark 2.2. It is not too hard within the context of Schauder theory to reduce the assumptions on u in the above result to $u\in C^0(\overline{\Omega})\cap H^1(\Omega)$. See the HW problem #8 for more on this. A much deeper result is that we only need to assume a weak $H^1(\Omega)$ solution is bounded to obtain full regularity. But this latter result goes beyond the limitations of Schauder theory.

Proof. As usual, we may subtract off an extension $\widetilde{g}$ of the boundary data at the price of adding an additional term:

$$F = -\sum_{i,j} \partial_i \big[A_{ij}(u) \partial_j \widetilde{g} \big] ,$$

to the RHS of (16). In doing so, we trade u on the LHS with $\widetilde{u} = u - \widetilde{g}$ which is in $H^1_0(\Omega)$.

Since we already assume a pointwise bound on ∇u, the RHS of the modified version of (16) is bounded, and the coefficient matrix $A(u)$ has $C^\alpha(\overline{\Omega})$ argument for all $\alpha \in (0,1)$. Thus, by the first divergence form Schauder estimate, we must in fact have that $\widetilde{u} \in C^{1,\alpha}(\overline{\Omega})$ for all $\alpha \in (0,1)$, which implies the same estimate for u because $\widetilde{g}$ is assumed to be smooth. This in turn means the the RHS of (16) is in fact in $C^\alpha(\overline{\Omega})$, while the coefficient matrix $A_{ij}(u)$ is in $C^{1,\alpha}(\overline{\Omega})$. By the higher regularity form of the weak Schauder estimate, we then have $\widetilde{u} \in C^{2,\alpha}(\overline{\Omega})$, which again gives the same estimate for u.

Clearly this process can be repeated inductively to show that $u \in C^{k,\alpha}(\overline{\Omega})$ for all k. □

Department of Mathematics, University of California, San Diego (UCSD) 9500 Gilman Drive, Dept 0112 La Jolla, CA 92093-0112 USA
E-mail address: jsterben@math.ucsd.edu