

Solutions for Homework 1

Problem 1 (6.1.5). Solve $u_{xx} + u_{yy} = 1$ in $r < a$ with $u(x, y)$ vanishing on $r = a$.

Solution. Since the data of this problem (that is, the right hand side and the boundary conditions) are all radially symmetric, it makes sense to try for a solution which is also radially symmetric. Therefore we assume that, in polar coordinates, u is a function of the radius r exclusively, in hopes of getting a solvable ODE.

The laplacian in polar coordinates is given by $\frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r}$. (We omit the θ term since we assume u depends only on r) Therefore we have $u_{rr} + (1/r)u_r = 1$ for $0 \leq r < a$ and $u = 0$ for $r = a$. If we observe that $1/r(ru_r)_r = u_{rr} + (1/r)u_r$, we solve the ODE by integration:

$$\begin{aligned} (ru_r)_r = r &\Rightarrow ru_r = \frac{r^2}{2} + C_1 \\ \Rightarrow u_r &= \frac{r}{2} + \frac{C_1}{r} \\ \Rightarrow u &= \frac{r^2}{4} + C_1 \ln r + C_2 \end{aligned} \tag{1}$$

Since u must be continuous on the entire unit disc, $C_1 = 0$, and the boundary condition forces $C_2 = -a^2/4$. Therefore our answer is

$$u(r) = \frac{r^2 - a^2}{4} \tag{2}$$

By the uniqueness theorem for the Dirichlet problem, this must be the only solution.

Problem 2 (6.1.10). Prove the uniqueness of the Dirichlet problem $\Delta u = f$ in D , $u = g$ on ∂D by the energy method. That is, after subtracting two solutions $w = u - v$, multiply the Laplace equation for w by w itself and use the divergence theorem.

Solution. Taking the given hint, we integrate the expression $w\Delta w$ over our spatial domain Ω and use the fact that

$$\nabla \cdot (w\nabla w) = |\nabla w|^2 + w\Delta w. \tag{3}$$

Since $w = u - v$, the equation we must solve is $\Delta w = 0$ on Ω and $w = 0$ on $\partial\Omega$. If we can show that $w = 0$ is the only solution, then we have proven

uniqueness. So,

$$\begin{aligned} 0 &= \iiint_{\Omega} w\Delta w \, dV \\ &= - \iiint_{\Omega} |\nabla w|^2 \, dV + \iiint_{\Omega} \nabla \cdot (w\nabla w) \, dV \\ &= - \iiint_{\Omega} |\nabla w|^2 \, dV + \iint_{\partial\Omega} w(\nabla w \cdot \mathbf{n}) \, dS \\ &= - \iiint_{\Omega} |\nabla w|^2 \, dV \end{aligned} \tag{4}$$

The last equality comes from the fact that $w = 0$ on $\partial\Omega$.

Now, $|\nabla w|^2 \geq 0$, so (4) requires that $\nabla w = 0$ on Ω . Therefore w is a constant function. But we also know that $w = 0$ on $\partial\Omega$. Thus we conclude that $u - v = w = 0$ everywhere on Ω , hence the solution is unique.

Problem 3 (7.1.1). Derive the three-dimensional maximum principle from the mean value property.

Solution. We note first that, as with all averages, we have

$$\frac{1}{A(\partial D)} \int_{\partial D} u \, dS \leq \max u \tag{5}$$

with equality if and only if $u(\mathbf{x}) = \max u$ on the entire surface ∂D . That is, the average is less than the max of a function, and they are equal only in the case when everything being averaged is the same.

Now, suppose \mathbf{x} is an interior maximizer for u , that is, $u(\mathbf{x}) \geq u(\mathbf{y})$ for all $\mathbf{y} \in D$. I claim that u must be a constant function. We first show this in a ball surrounding \mathbf{x} . Let B be a ball with \mathbf{x} at its center, small enough to fit in D . Then since $u(\mathbf{x}) \geq u(\mathbf{y})$ for all \mathbf{y} on the surface of the ball, by the mean value property and (5), $u(\mathbf{y}) = u(\mathbf{x})$ for all \mathbf{y} in the surface of the ball. But notice that this is also true for any sphere centered at \mathbf{x} and contained in the original ball B , hence u is constant throughout B .

We can extend this argument to all of D . Pick any point \mathbf{y} in D . Then connect \mathbf{x} and \mathbf{y} by a path (we assume this is possible, that is, we assume D is *path connected*) and cover this path with overlapping balls.¹ Then u must be constant over the entire path, and hence $u(\mathbf{x}) = u(\mathbf{y})$. Since \mathbf{y} was arbitrary, this is true for all points of D , and hence u is a constant function.

¹This is always possible. For those who have taken analysis, this is because the path is a *compact* set, while the boundary of the ball is a closed set, and a theorem of analysis says that the distance between two disjoint sets, one of which is closed and one compact, is always strictly positive. Make the radius of each ball smaller than this distance for the proof to work.

Problem 4 (7.1.3). *Prove the uniqueness of the Robin problem $\partial u/\partial n + a(\mathbf{x})u(\mathbf{x}) = h(\mathbf{x})$ provided that $a(\mathbf{x}) > 0$ on the boundary.*

Solution. Suppose u and v are two solutions to the Robin problem, and let $w = u - v$. Then $\Delta w = 0$ on D and $\partial w/\partial n + a(\mathbf{x})w(\mathbf{x}) = 0$ on S . In order to apply the energy method, we need to look at the quantity $w\Delta w$. Clearly, this is zero everywhere on D , if we integrate over D we get:

$$0 = \int_D w\Delta w \, dV = - \int_D |\nabla w|^2 \, dV + \int_S w \frac{\partial w}{\partial n} \, dS \quad (6)$$

Now, if we multiply the boundary equation by w , we get:

$$w \frac{\partial w}{\partial n} + a(\mathbf{x})w^2(\mathbf{x}) = 0. \quad (7)$$

Combining equations (6) and (7), we find that

$$\int_D |\nabla w|^2 \, dV = - \int_S a(\mathbf{x})w^2(\mathbf{x}) \, dS. \quad (8)$$

The left hand side of (8) is nonnegative, the right hand side is nonpositive, hence both sides are zero. Therefore $w(\mathbf{x}) = 0$ on S , and $\nabla w(\mathbf{x}) = 0$ on D ; these two together imply that $w(\mathbf{x}) = 0$ on D , and hence that $u(\mathbf{x}) = v(\mathbf{x})$.

Problem 5 (7.1.9). *Consider the problem $u_{xx} + u_{yy} = 0$ in the triangle $\{x > 0, y > 0, 3x + y < 3\}$ with the boundary conditions*

$$u(x, 0) = 0 \quad u(0, y) = y(3 - y) \quad u(x, 3 - 3x) = 0$$

Choose $w_0 = y(3 - 3x - y)$, $w_1 = xy(3 - 3x - y)$, and $w_2 = x^2y(3 - 3x - y)$. Find the Rayleigh–Ritz approximation of the form $w_0 + c_1w_1 + c_2w_2$ to u .

Solution. From problem 7.1.7, we know that we must solve the linear system

$$\sum_{k=1}^2 (\nabla w_j, \nabla w_k) c_k = -(\nabla w_0, \nabla w_j) \quad \text{for } j = 1, 2 \quad (9)$$

Where parentheses indicate we must integrate over the domain D . If we write this as a matrix equation, we get

$$\begin{bmatrix} (\nabla w_1, \nabla w_1) & (\nabla w_1, \nabla w_2) \\ (\nabla w_2, \nabla w_1) & (\nabla w_2, \nabla w_2) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = - \begin{bmatrix} (\nabla w_0, \nabla w_1) \\ (\nabla w_0, \nabla w_2) \end{bmatrix} \quad (10)$$

Unfortunately, these integrals are quite tedious to compute. You might notice that if you apply Green's first identity, the boundary term is in all cases zero, as long as the correct choice is made applying the identity. However, this doesn't make the problem much easier.

The computations should come out as follows:

$$\begin{bmatrix} \frac{3}{2} & \frac{9}{20} \\ \frac{9}{20} & \frac{9}{40} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{9}{20} \\ -\frac{3}{20} \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{4} \\ -\frac{1}{6} \end{bmatrix} \quad (11)$$

Finally, we compute²

$$w = w_0 + c_1w_1 + c_2w_2 = \frac{y}{12}(2x^2 + 3x - 12)(3x + y - 3) \quad (12)$$

Problem 6 (7.2.3). *Give yet another derivation of the mean value property in three dimensions by choosing D to be a ball and \mathbf{x}_0 its center in the representation formula*

$$u(\mathbf{x}_0) = \iint_{\partial D} \left[-u(\mathbf{x}) \frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \right) + \frac{1}{|\mathbf{x} - \mathbf{x}_0|} \frac{\partial u}{\partial n} \right] \frac{dS}{4\pi} \quad (13)$$

Solution. Let D be a ball of radius r centered at \mathbf{x}_0 . We look at the second term of (13) first. Since the radius vector is constant on the surface of a sphere, we can pull it out of the integral. Then we can apply the divergence theorem as follows:

$$\begin{aligned} \iint_{\partial D} \frac{1}{|\mathbf{x} - \mathbf{x}_0|} \frac{\partial u}{\partial n} \frac{dS}{4\pi} &= \frac{1}{r} \iint_{\partial D} \frac{\partial u}{\partial n} \frac{dS}{4\pi} \\ &= \frac{1}{r} \iiint_D \Delta u \frac{dV}{4\pi} \\ &= 0 \end{aligned} \quad (14)$$

For this reason we need only work with the first term of (13). Now,

$$\frac{\partial}{\partial n} \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \right) = \nabla \left(\frac{1}{|\mathbf{x} - \mathbf{x}_0|} \right) \cdot \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} = -\frac{1}{|\mathbf{x} - \mathbf{x}_0|^2} \quad (15)$$

When we plug this back in to (13), we get

$$u(\mathbf{x}_0) = \frac{1}{4\pi r^2} \iint_{\partial D} u(\mathbf{x}) \, dS \quad (16)$$

and this is our original mean value formula.

²The solution in the text is given as $c_1 = -0.248$ and $c_2 = -0.008$. However, when we compute the energy itself for the two solutions, the book's solution gives 22.3685 whereas this solution gives 22.3625, so the book's answer appears to be incorrect.