

Solutions for Homework 2

Problem 1 (7.4.5). Notice that the function xy is harmonic in the half-plane $\{y > 0\}$ and vanishes on the boundary line $\{y = 0\}$. The function 0 has the same properties. Does this mean that the solution is not unique? Explain.

Solution. In order to prove uniqueness for the Dirichlet problem on an unbounded (in any direction) domain, we must include the condition that the limit of the function at infinity is zero; in other words, the “boundary” value at infinity is zero. Clearly, xy does not meet this condition.

Problem 2 (7.4.8). 1. Use Exercise 7 to find the harmonic function in the half-plane $\{y > 0\}$ with the boundary data $h(x) = 1$ for $x > 0$, $h(x) = 0$ for $x < 0$.

2. Do the same as part 1 for the boundary data $h(x) = 1$ for $x > a$, $h(x) = 0$ for $x < a$ (Hint: Translate the preceding answer.)

3. Use part 2 to solve the same problem with the boundary data $h(x)$ where $h(x)$ is any step function. That is,

$$h(x) = c_j \quad \text{for } a_{j-1} < x < j \quad \text{for } 1 \leq j \leq n$$

where $-\infty = a_0 < a_1 < \dots < a_{n-1} < a_n = \infty$ and the c_j are constants.

Solution. From the previous problem, we are given a harmonic function in the half-plane with as yet undetermined boundary conditions, namely

$$u(x, y) = \begin{cases} c_1 \arctan\left(\frac{x}{y}\right) + c_2 & \text{for } y > 0 \\ c_1 \frac{\pi}{2} + c_2 & \text{for } y = 0, x > 0 \\ 0 & \text{for } y = 0, x = 0 \\ -c_1 \frac{\pi}{2} + c_2 & \text{for } y = 0, x < 0 \end{cases} \quad (1)$$

Clearly, these boundary conditions will suit us if we solve for c_1 and c_2 , with the value of 1 for $x > 0$ and 0 for $x < 0$. The values we get are $c_1 = 1/\pi$, $c_2 = 1/2$. Therefore

$$u(x, y) = \frac{1}{\pi} \arctan\left(\frac{x}{y}\right) + \frac{1}{2} \quad (2)$$

For the second part, we wish to translate this solution in the x direction by a . In general, $f(x - a, y - b)$ is a translation of the function $f(x, y)$ by the

constant vector (a, b) , so our new solution becomes:

$$u_{(a, \infty)}(x, y) = \frac{1}{\pi} \arctan\left(\frac{x - a}{y}\right) + \frac{1}{2} \quad (3)$$

This is the definition of $u_{(a, \infty)}(x, y)$. This notation indicates that the function is one on the interval (a, ∞) , zero on $(-\infty, a)$. Similarly, we can define

$$u_{(-\infty, a)}(x, y) = \frac{1}{\pi} \arctan\left(\frac{a - x}{y}\right) + \frac{1}{2}. \quad (4)$$

This function is just the reflection of (3) about the y axis.

In keeping with equations (3) and (4), we can express $u_{(a, b)}$ for any interval. For example, if we want a function which has a boundary value of 1 in the interval $(4, 9)$, zero outside $[4, 9]$, we just subtract:

$$u_{(4, 9)}(x, y) = (u_{(4, \infty)}(x, y) - u_{(9, \infty)}(x, y)) \quad (5)$$

With this in mind, we define

$$u_{(a, b)}(x, y) = (u_{(a, \infty)}(x, y) - u_{(b, \infty)}(x, y)). \quad (6)$$

We now have consistent notation for any interval, so we take linear combinations of these functions to get the desired boundary conditions:

$$u(x, y) = \sum_{j=0}^{n-1} c_j u_{(a_j, a_{j+1})}(x, y). \quad (7)$$

Notice that we use the linearity of the laplace operator.

Problem 3 (7.4.12). Find the potential of the electrostatic field due to a point charge located outside a grounded sphere. (Hint: This is just the Green’s function for the exterior of the sphere. Find it by the method of reflection.)

Solution. We are given the Green’s function for a sphere, namely:

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi\rho} + \frac{a}{|\mathbf{x}_0|} \frac{1}{4\pi\rho^*} \quad (8)$$

where a is the radius of the sphere, $\rho = |\mathbf{x} - \mathbf{x}_0|$, and $\rho^* = |\mathbf{x} - \mathbf{x}_0^*|$. Now, recall that a Green’s function must meet three criteria:

(i) $\Delta G = 0$, except at \mathbf{x}_0

(ii) $G(\mathbf{x}) = 0$ on the boundary

(iii) $G(\mathbf{x}) + 1/(4\pi|\mathbf{x} - \mathbf{x}_0|)$ is continuous and harmonic on the entire domain

What is our domain in this problem? Clearly, it is the complement to the ball; that is, it is everything that is not in the ball. This region's boundary is the surface of the sphere, plus the point at infinity. Clearly the function given in (8) is zero at these boundaries. Also, it is clearly harmonic everywhere *except* at \mathbf{x}_0^* . It therefore makes sense to consider interchanging the roles of \mathbf{x}_0 and \mathbf{x}_0^* . In that case, our candidate for Green's function looks like

$$G(\mathbf{x}, \mathbf{x}_0) = -\frac{1}{4\pi\rho^*} + \frac{a}{|\mathbf{x}_0^*|} \frac{1}{4\pi\rho} \quad (9)$$

where \mathbf{x}_0 is now a point on the *outside* of the sphere.

We might wish to stop here; however, we have not yet met the third condition of a Green's function. Therefore we must multiply the whole thing by $-\frac{|\mathbf{x}_0^*|}{a}$. Our final answer is now

$$G(\mathbf{x}, \mathbf{x}_0) = \frac{|\mathbf{x}_0^*|}{a} \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0^*|} - \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_0|} \quad (10)$$

where \mathbf{x}_0 is a point in our domain; that is, outside the sphere, and \mathbf{x}_0^* is its reflection on the inside of the ball, as given in the text.

Problem 4 (7.4.14). *Find the Green's function for the eighth of a ball*

$$D = \{x^2 + y^2 + z^2 < a^2, x > 0, y > 0, z > 0\}. \quad (11)$$

Solution. Prior to dealing with the problem, we must determine a convention for naming the eight octants of three dimensional cartesian space. I have given the conventions I use in table 1.

For this, we start with the Green's function for the half-space, which is given in the text and which we will call $H(\mathbf{x}, \mathbf{x}_0)$. Since we are going to wind up with a reflected point in each of the eight octants, we will denote these by $\mathbf{x}_1, \dots, \mathbf{x}_8$, where the index denotes the octant in which the point resides. Also, there will be the corresponding starred points, $\mathbf{x}_1^*, \dots, \mathbf{x}_8^*$. Since we are trying to find the Green's function for the first octant, we would be more consistent to write $H(\mathbf{x}, \mathbf{x}_1)$.

First, let us reflect across the xy plane. Since this reflection yields the point $(x, y, -z)$, we consult table 1 to obtain the fact that our reflected point will correspond to the eighth octant. We denote the reflected point

\mathbf{x}_i	octant number	sign pattern	σ
(x, y, z)	1	$(+, +, +)$	+
$(-x, y, z)$	2	$(-, +, +)$	-
$(-x, -y, z)$	3	$(-, -, +)$	+
$(x, -y, z)$	4	$(+, -, +)$	-
$(x, -y, -z)$	5	$(+, -, -)$	+
$(-x, -y, -z)$	6	$(-, -, -)$	-
$(-x, y, -z)$	7	$(-, +, -)$	+
$(x, y, -z)$	8	$(+, +, -)$	-

Table 1: The octants of cartesian space. σ denotes the sign of the octant.

in octant 8 by \mathbf{x}_8 . Let us call the Green's function for this half ball $K(\mathbf{x}, \mathbf{x}_1)$, then

$$K(\mathbf{x}, \mathbf{x}_1) = H(\mathbf{x}, \mathbf{x}_1) - H(\mathbf{x}, \mathbf{x}_8) \quad (12)$$

To verify that K is in fact the Green's function for the half-sphere, just check the three criteria. Only the boundary condition criterion (number 2 in the text) gives any trouble. But the reason we subtract $H(\mathbf{x}, \mathbf{x}_8)$ in (12) is to zero out the boundary condition.

Now we repeat this procedure twice to get down to the quarter-ball, and then to the eighth-ball. First, for the quarter ball, we reflect across the yz plane. Consult table 1 again to conclude that \mathbf{x}_2 is the new reflection point. If we denote the Green's function for the quarter-ball by $L(\mathbf{x}, \mathbf{x}_1)$, then

$$L(\mathbf{x}, \mathbf{x}_1) = K(\mathbf{x}, \mathbf{x}_1) - K(\mathbf{x}, \mathbf{x}_2). \quad (13)$$

Finally, for the eighth-ball, we call the Green's function G , and we get

$$G(\mathbf{x}, \mathbf{x}_1) = L(\mathbf{x}, \mathbf{x}_1) - L(\mathbf{x}, \mathbf{x}_4). \quad (14)$$

You may notice that we have only used three points in octants other than the first, rather than one point in each. However, since our definition of each new Green's function was recursive, all the points are included implicitly; for example, in equation (13), the second term $K(\mathbf{x}, \mathbf{x}_2) = H(\mathbf{x}, \mathbf{x}_2) - H(\mathbf{x}, \mathbf{x}_6)$.

Now, with all that recursion, we get exactly one copy of Green's function for the entire sphere for each octant. But with what sign? This is where the sign of the octant comes in handy; I assert that the sign of each term is, in fact, the sign of its respective octant. You should check this yourself; it should be clear, however, that this property will always hold when using the reflection principle.

We can now write the solution in a single formula:

$$G(\mathbf{x}, \mathbf{x}_1) = \sum_{i=1}^8 \sigma(i)H(\mathbf{x}, \mathbf{x}_i) \quad (15)$$

where H is the Green's function for the whole ball.

Problem 5 (7.4.16). *Use exercises 15 and 7 to find the harmonic function $u(x, y)$ in the first quadrant that has the boundary values $u(x, 0) = A$, $u(0, y) = B$, where A and B are constants. (Hint: $u(x, 0) = v(x^2, 0)$, etc.)*

Solution. Let us first define two cartesian coordinate systems: the (x, y) plane, where our "first quadrant" region lives, and (w, z) , where our "half-plane" region lives. Then problem 15 gives us a transformation $\Phi(x, y) \mapsto (w, z)$ that does three things:

1. It transforms the first quadrant into the half-plane $\{y > 0\}$.
2. For an harmonic function $v(w, z)$ defined on the half-plane, the function $v(\Phi(x, y))$ is harmonic in the first quadrant.
3. Though not stated in problem 15, it should be clear that Φ maps the positive x axis to the positive w axis, and the positive y axis to the negative w axis.

Given these three facts, any function v which meets some boundary conditions on the half-plane will, under the transformation Φ , meet those same boundary conditions (and retain the property of being harmonic) in the first quadrant.

Problem 7 gives us such a function. Solving for these particular boundary conditions, we get

$$v(w, z) = \frac{A - B}{\pi} \arctan\left(\frac{w}{z}\right) + \frac{A + B}{2} \quad (16)$$

Plugging in our given transformation Φ gives

$$u(x, y) = v(\Phi(x, y)) = \frac{A - B}{\pi} \arctan\left(\frac{x^2 - y^2}{2xy}\right) + \frac{A + B}{2}. \quad (17)$$

Problem 6 (7.4.18).

- (a) *Find the Green's function for the octant $\mathcal{O} = \{(x, y, z) : x > 0, y > 0, z > 0\}$. (Hint: Use the method of reflection.)*

(b) *Use your answer in (a) to solve the Dirichlet problem*

$$\begin{cases} u_{xx} + u_{yy} + u_{zz} = 0 & \text{in } \mathcal{O} \\ u(0, y, z) = 0, u(x, 0, z) = 0, u(x, y, 0) = h(x, y) & \text{for } x, y, z > 0 \end{cases}$$

Solution. We use the same idea as in problem 14. To find the Green's function on the half-plane, for example, we used $\mathbf{x}_1 = (x_1, y_1, z_1)$ and $\mathbf{x}_8 = (x_1, y_1, -z_1)$ as a reflected pair, so our Green's function, denoted by H here, became

$$H(\mathbf{x}, \mathbf{x}_1) = -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_1|} + \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_8|} \quad (18)$$

As in problem 14, we use table 1 to determine which octant to choose our reflected point from.

To get the Green's function for the quarter-space, just reflect across one of the remaining planes, for example the yz plane. Then, denoting this Green's function by K , and defining $\mathbf{x}_2 = (-x_1, y_1, z_1)$, $\mathbf{x}_7 = (-x_1, y_1, -z_1)$, we get

$$\begin{aligned} K(\mathbf{x}, \mathbf{x}_1) &= H(\mathbf{x}, \mathbf{x}_1) - H(\mathbf{x}, \mathbf{x}_2) \\ &= -\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_1|} + \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_8|} \\ &\quad + \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_2|} - \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_7|} \end{aligned} \quad (19)$$

Repeat this step one more time, and we get our Green's function G , which looks like:

$$G(\mathbf{x}, \mathbf{x}_1) = \sum_{i=1}^8 -\sigma(i) \frac{1}{4\pi|\mathbf{x} - \mathbf{x}_i|}. \quad (20)$$

We follow again the conventions of table 1.

For part (b), we must take the normal derivative of this function and integrate over the boundary. Fortunately, only one boundary condition is non-zero, and the normal derivative here (the xy plane) is just $-\partial/\partial z$. If we look at one prototypical term from (20), the normal derivative is

$$-\frac{\partial}{\partial z} \left(\frac{1}{4\pi|\mathbf{x} - \mathbf{x}_i|} \right) = \frac{z - z_i}{4\pi|\mathbf{x} - \mathbf{x}_i|^3}. \quad (21)$$

Note also that since the boundary condition is nonzero on the xy plane, the denominators in the terms of (20) are equal for terms that are paired across the xy plane. Thus octants 1 and 8, 2 and 7, 3 and 6, 4 and 5, are

naturally paired. Their z components are of opposite sign, however; that is, $z_1 = -z_8$, etc. Keeping careful track of signs allows us to conclude that

$$\frac{\partial G}{\partial n} = \sum_{i=1}^4 -\sigma(i) \frac{2z_1}{4\pi|\mathbf{x} - \mathbf{x}_i|^3} \quad (22)$$

and therefore

$$u((x_1, y_1, z_1)) = \frac{z_1}{2\pi} \sum_{i=1}^4 -\sigma(i) \int_0^\infty \int_0^\infty \frac{h(x, y)}{((x - x_i)^2 + (y - y_i)^2 + z_i^2)^{\frac{3}{2}}} dx dy. \quad (23)$$

Problem 7 (7.4.23). *Solve the Neumann problem in the quarter-plane $\{x > 0, y > 0\}$.*

Solution. The Neumann problem specifies the outward normal derivative on the boundary. Problem 21 gives us a criterion for the Neumann function $N(x, y)$, which is analogous to a Green's function. In particular, we are looking for a function that satisfies the following:

- (i) N is continuous, with continuous second derivatives and $\Delta N = 0$, except at a point $\mathbf{x} = \mathbf{x}_0$.
- (ii) $\frac{\partial N}{\partial n} = c$ on the boundary
- (iii) $N(x, y) + \frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_0|)$ is continuous, with continuous second derivatives, on the entire domain.

Furthermore, problem 21 tells us that $c = 0$ when the boundary is infinite.

We proceed as before by first computing the half-plane formula, then the quarter-plane. For the half-plane, we reflect over the x axis, so that if $\mathbf{x}_1 = (x_1, y_1)$ (the point for which we will solve) then $\mathbf{x}_4 = (x_1, -y_1)$ is the reflected point, and

$$H(\mathbf{x}, \mathbf{x}_1) = - \left[\frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_1|) + \frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_4|) \right] \quad (24)$$

You should check that this function meets the three criteria for a Neumann function, particularly the second condition. Notice also that the terms do not alternate sign, as in the Dirichlet problem.

We can reflect across the y axis, picking up two more reflected points. In this case, we define $\mathbf{x}_3 = (-x_1, -y_1)$, and $\mathbf{x}_2 = (-x_1, y_1)$. Then \mathbf{x}_2 is the

reflection of \mathbf{x}_1 across the y axis, and \mathbf{x}_3 is the reflection of \mathbf{x}_4 across the y axis. Our final Neumann function is given by

$$\begin{aligned} N(\mathbf{x}, \mathbf{x}_1) &= H(\mathbf{x}, \mathbf{x}_1) + H(\mathbf{x}, \mathbf{x}_2) \\ &= - \left[\frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_1|) + \frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_4|) \right. \\ &\quad \left. + \frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_2|) + \frac{1}{2\pi} \ln(|\mathbf{x} - \mathbf{x}_3|) \right] \\ &= - \frac{1}{2\pi} \sum_{i=1}^4 \ln(|\mathbf{x} - \mathbf{x}_i|). \end{aligned} \quad (25)$$

The Neumann problem requires that the normal derivative be specified on the boundaries; let these be given by $h(x)$ and $k(y)$ on the x and y axes, respectively. Now that we have the Neumann function, the problem is solved by applying the integration formula, analogous to the solution to the Dirichlet problem with a Green's function. This formula was derived in problem 21,

$$\begin{aligned} u(\mathbf{x}_1) &= - \int_{\partial D} N(\mathbf{x}, \mathbf{x}_1) \frac{\partial u(\mathbf{x})}{\partial n} ds. \\ &= \int_0^\infty h(x) \sum_{i=1}^4 \frac{1}{2\pi} \ln \left(\sqrt{(x - x_i)^2 + y_i^2} \right) dx \\ &\quad + \int_0^\infty k(y) \sum_{i=1}^4 \frac{1}{2\pi} \ln \left(\sqrt{x_i^2 + (y - y_i)^2} \right) dy \\ &= \frac{1}{4\pi} \sum_{i=1}^4 \left[\int_0^\infty h(x) \ln((x - x_i)^2 + y_i^2) dx \right. \\ &\quad \left. + \int_0^\infty k(y) \ln(x_i^2 + (y - y_i)^2) dy \right] \end{aligned} \quad (26)$$