MATH 132A
NOTES I

Abstract. Here is a set of sparse notes which contains material that you should know for the first exam.

1. Introduction

In these notes we list some notation, ideas, and formulas that you should be familiar with ahead of the first exam. There is certainly more material here than can be represented in exam problems (there will be only about 3-4), and the reader should note certain sections which are labeled as material *that will not be covered on the first exam*. Specifically, this includes Sections 4, 11, and 15, whose main purpose is to give some additional context to the core material.

Please make some effort to read these notes over before the exam. *You may bring these notes as well as your text as a reference for use during the exam period.*

2. A Little Notation

Lets begin by outlining some (but perhaps not all!) of the notation we intend to use here. First of all, we will always be working in 2 and 3 spatial dimensions. That is, we will be working on the spaces $\mathbb{R}^2$ and $\mathbb{R}^3$, which consist of all ordered pairs and triples of real numbers. For example:

$$(x, y) \in \mathbb{R}^2 , \qquad\qquad (x, y, z) \in \mathbb{R}^3 .$$

Here the "$\in$" notation just means that the ordered pair or triple in question is a single point in 2 or 3 dimensional space. At times it will be useful to denote a single point by $\vec{x}$ in either $\mathbb{R}^2$ or $\mathbb{R}^3$ depending on the context. The vector notation is meant to remind you that this is not simply a real number, but rather a vector with several components.

Depending on the context, we will use the following notations for partial derivatives (depending also on the dimension):

$$\partial_x u(x, y) = u_x(x, y) , \qquad \partial_y u(x, y) = u_y(x, y) ,$$
$$\partial_x v(x, y, z) = v_x(x, y, z) , \quad \partial_y v(x, y, z) = v_y(x, y, z) , \quad \partial_z v(x, y, z) = v_z(x, y, z) .$$

Of course, partial differentiation gives us the instantaneous rate of change of the function with respect to that variable (i.e. ∂_x for the x variable), when all the other variables are held fixed.

The main equation that we are studying right now is the *Laplace Equation*. This is defined as follows:

$$\begin{aligned} \Delta u &= \partial_x^2 u + \partial_y^2 u \ , && \text{on } \mathbb{R}^2 \ , \\ \Delta v &= \partial_x^2 v + \partial_y^2 v + \partial_z^2 v \ , && \text{on } \mathbb{R}^3 \ . \end{aligned}$$

At times, it is very useful to exploit the relationship between the Laplace equation and the formulas which define divergence free, irrotational vector-fields. To discuss this, lets first set up the usual vector notation. For functions u and v, sometimes called *scalars*, we can form a vector-field as follows:

$$\begin{aligned} \nabla u &= (u_x, u_y) \ , && \text{on } \mathbb{R}^2 \ , \\ \nabla v &= (v_x, v_y, v_z) \ , && \text{on } \mathbb{R}^3 \ . \end{aligned}$$

This "∇" is often referred to as the *gradient*, and sometimes one uses *grad* instead of ∇.

Recall that a vector-field $\vec{F}$ is simply an ordered collection of functions, one for each coordinate direction in the underlying space. So, for example we have:

$$\begin{aligned} \vec{F} &= \big(\vec{F}^x(x,y), \vec{F}^y(x,y)\big) \ , && \text{on } \mathbb{R}^2 \ , \\ \vec{G} &= \big(\vec{G}^x(x,y,z), \vec{G}^y(x,y,z), \vec{G}^z(x,y,z)\big) \ , && \text{on } \mathbb{R}^3 \ . \end{aligned}$$

Here I have used the superscript notation to denote each component of the vector-field. This is to avoid confusion with the earlier subscript notation for partial derivatives. The key thing to understand is that each component, for example $\vec{F}^y(x,y)$, is a single real valued (scalar) function.

The most common way to differentiate vector-fields (at least in this class) is to take their *divergence*. This is defined as follows:

$$\begin{aligned} \nabla \cdot \vec{F} &= \partial_x(\vec{F}^x) + \partial_y(\vec{F}^y) \ , && \text{on } \mathbb{R}^2 \ , \\ \nabla \cdot \vec{G} &= \partial_x(\vec{G}^x) + \partial_y(\vec{G}^y) + \partial_z(\vec{G}^z) \ , && \text{on } \mathbb{R}^3 \ . \end{aligned}$$

The key thing to notice here is that once you form the divergence, you end up with a scalar. That is, the divergence takes a vector-field and gives you back a single function. Roughly speaking, the divergence produces a local density which measures the amount of "sources" and "sinks" in the flow of the vector-fields $\vec{F}$ or $\vec{G}$. See Appendix A for a bit more on this. At times, the divergence is also written as $div\,\vec{F}$ instead of $\nabla \cdot \vec{F}$.

Using the gradient and divergence operators, we can now write the Laplace equation in the following form (this works equally well on $\mathbb{R}^2$ and $\mathbb{R}^3$):

$$\Delta u \ = \ \nabla \cdot (\nabla u) \ = \ div(grad\, u) \ .$$

2.1. **Some Additional Notation: Inner Products.** We record here some additional notation that comes up in the text (for instance in Problem 7 on p. 175). Often we will be concerned with certain inner produces of vectors. For example, if u and v are two functions on $\mathbb{R}^3$ we can form the product of their gradients:

$$\nabla u \cdot \nabla v \ = \ u_x v_x + u_y v_y + u_z v_z \ . \tag{1}$$

Notice that in this case, the "inner product" produces a scalar function that can vary from point to point in some domain $\Omega \subseteq \mathbb{R}^3$. For many purposes, it is useful to define inner products that give a single (constant) real number depending on the input. One example is the standard inner product between two *real* valued functions f and g (if they are complex valued, you need to substitute the complex conjugate $\overline{g}$ on the R.H.S below):

$$\langle f, g \rangle \ = \ \int_\Omega fg \ d\vec{x} \ .$$

Here the domain Ω could be in $\mathbb{R}^2$ or $\mathbb{R}^3$, or even some closed interval $[a, b] \subseteq \mathbb{R}$. This inner product produces a single real number that depends on f and g, and in some sense it measures to what extent f and g represent distinct information. For example, if $\langle f, g \rangle = 0$, then there is no "overlapping" between f and g and they represent completely distinct pieces of information. On the other hand, if $\langle f, g \rangle = \langle f, f \rangle^{\frac{1}{2}} \langle g, g \rangle^{\frac{1}{2}}$, then it can be shown that $g = cf$ for some constant c. That is, in this case f and g are essentially the same functions.

Another useful inner product is to integrate the quantity $\nabla u \cdot \nabla v$ from line (1) above over the domain Ω:

$$\langle \nabla u, \nabla v \rangle \ = \ \int_\Omega \nabla u \cdot \nabla v \ d\vec{x} \ .$$

This is a common expression that comes up when one computes the "energy" of linear combinations of functions (see Section 15 below).

3. The Main Problems

Everything that follows in these notes concerns solving the equation $\Delta u = f$ in various circumstances. Even if one only wishes to study the equation $\Delta u = 0$, it turns out that to pin down specific solutions more information is required. This is because there are *many* solutions to $\Delta u = 0$. We call such solutions u *harmonic functions*. These functions turn out to have remarkable properties, and we will be able to discuss only a few of these properties in this class.

However, for many of the physically interesting situations we are interested in, a more general set of problems is appropriate. All of these lead to specific (unique) solutions as soon as we specify certain *boundary conditions*. That is, from a purely mathematical perspective one can think of specifying boundary conditions as a way to pin down specific solutions to Laplace's equation. From a physical perspective, one can think of specific boundary conditions as input needed to fully describe the physical situation (e.g. a temperature distribution on the boundary of some solid object is needed to know the corresponding steady state distribution inside the body).

In what follows, Ω is some bounded (i.e. it has compact closure) domain in $\mathbb{R}^2$ or $\mathbb{R}^3$ with a nice (say smooth) boundary. The main problems we study are the following:

$$\Delta u = f , \quad \text{in } \Omega , \tag{2a}$$
$$u|_{\partial\Omega} = g . \qquad \text{(Dirichlet condition)} \tag{2b}$$

$$\Delta u = f , \quad \text{in } \Omega , \tag{2c}$$
$$\hat{n}\cdot\nabla u|_{\partial\Omega} = h . \qquad \text{(Neumann condition)} \tag{2d}$$

$$\Delta u = f , \quad \text{in } \Omega , \tag{2e}$$
$$\big(u + a(\vec{x})\,\hat{n}\cdot\nabla u\big)|_{\partial\Omega} = h . \qquad \text{(Robin condition)} \tag{2f}$$

On line (2d) above, $\hat{n}$ denotes the outward unit normal to Ω. Also, on line (2f) $a(\vec{x})$ is some scalar function defined on $\partial\Omega$. For the most part, we will be dealing with the Dirichlet and Neumann problems. We'll discuss the Robin problem a little in Section 4 below. See also the HW.

4. Some Physical Motivation (**not required for Exam 1**)

In this section, we pause to give a little bit of motivation for the problems (2b)–(2f). All of these correspond to certain situations which occur naturally, and it may help you to have some "physical intuition" about the various phenomena which underlie the set of mathematical problems we discuss.

Lets start with the Dirichlet problem (2b). This is the most "canonical" boundary value problem. MORE LATER.

5. What We Would Like to Know

We would now like to make a list of the various questions that will concern us. One should not think of these as a list of "cookbook" problems that can be solved for the Laplace equation, but rather as a general set of guidelines one can refer to when thinking about a specific situation (e.g. a specific boundary value problem, with explicitly given boundary data). Furthermore, this list is by no means definitive, and many of the individual items are connected to each other. Here are some very basic things that we would like to know:

- Existence, Uniqueness, and Stability of solutions.
- Maximum Principles for solutions.
- Average Value identities for solutions.
- Explicit integral formulas for solutions.

- Other explicit ways of writing down solutions when the underlying domain is special (e.g. when $\Omega = B_r$ or $\Omega = D_r$, the ball or disk of radius r centered at the origin).
- Formulas for "Harmonic Polynomials".

In the remaining sections we will discuss these various aspects solving $\Delta u = f$ in more detail. In fact, for the most part we will restrict our attention to the case $f = 0$ with specified boundary data (e.g. Dirichlet or Neumann).

6. Explicit Solutions in Circular and Spherical Symmetry

Lets get started by solving explicitly the equation $\Delta u = 0$ in $2D$ or $3D$ under the assumption that the solution u does not depend on the angular variables. That is, we are assuming that $u = u(r)$ is purely a function of the radial variable:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2}\ , && \text{for } (x, y) \in \mathbb{R}^2\ , \\
r &= \sqrt{x^2 + y^2 + z^2}\ , && \text{for } (x, y, z) \in \mathbb{R}^3\ .
\end{aligned}
$$

As discussed in class, as well as on p. 153 of the text, the Laplace equation simplifies to the following ODE in this case (see also Appendix A below):

$$
\begin{aligned}
\Delta u &= \partial_r^2 u + \frac{1}{r}\partial_r u\ , && \text{in } \mathbb{R}^2\ , \\
\Delta u &= \partial_r^2 u + \frac{2}{r}\partial_r u\ , && \text{in } \mathbb{R}^3\ .
\end{aligned}
$$

In either case, the resulting ODE are easy to solve by hand, and we have the following result:

Theorem 6.1 (Rotationally Symmetric Harmonic Functions)**.** *Suppose that $u = u(r)$ is a rotationally symmetric function such that $\Delta u = 0$ away from $r = 0$ (this is important!). Then there exist constants A, B such that:*

$$u(r) = A\ln(r) + B\ , \qquad \textit{in } \mathbb{R}^2\ , \tag{3}$$

$$u(r) = \frac{A}{r} + B\ , \qquad \textit{in } \mathbb{R}^3\ . \tag{4}$$

Remark 6.2. In particular, note that the only rotationally symmetric harmonic functions which are regular when $r = 0$ are constants. This is because both of the functions listed above"blow up" as $r \to 0$ unless $A = 0$. There is a very good reason for this. If it were not the case, then one could find two distinct solutions to the Dirichlet problem from line (2b) above with $f = 0$, and $g = 0$ when $r = 1$ on $\mathbb{R}^3$. For example by setting $A = -B = 1$ on line (4), we have a solution to $\Delta u = 0$ that is also zero when $r = 1$. Now $u(r) \equiv 0$ is also a solution to $\Delta u = 0$ with this same boundary data. But in just a bit we will show that $u(r) \equiv 0$ is the *only* solution which extends continuously to $r = 0$. Therefore, we see that something *must* go wrong for any other spherical solution with this same boundary data, in such a way that we can't define things properly all throughout $r < 1$.

Example 6.3. Having the explicit formulas (3)–(4) allows us to write down solutions to the problems (2b)–(2f) with symmetric boundary data. Here is an example. Suppose that we with to solve the problem:

$$\begin{aligned} \Delta u &= 0 , \quad \text{when } 1 < r < 2 \text{ in } \mathbb{R}^2 , \\ u|_{r=1} &= 1 , \\ u|_{r=2} &= 2 . \end{aligned}$$

Here our domain Ω has two "boundary circles" given by $r = 1$ and $r = 2$, and it consists of all points in the region between these two circles (sometimes called an *annulus*). Now, from equation (3) above we know that there must be constants A and B such that (of course we are assuming that the solution u is rotationally symmetric, but since the boundary conditions are symmetric this is not a huge leap of faith):

$$u(r) = A \ln(r) + B .$$

To pin down A and B, we use the boundary conditions. When $r = 1$ we have that $u(1) = 1$, so we immediately get $B = 1$. Likewise, we also need that $u(2) = 2$, so we see that A must be given by $A = \frac{1}{\ln(2)}$. Thus, our explicit solution is:

$$u(r) = \frac{\ln(r)}{\ln(2)} + 1 .$$

7. Maximum Principles

We now uncover one the most remarkable properties of solutions to $\Delta u = 0$:

Theorem 7.1 (Strong Maximum Principle)**.** *Suppose that Ω is a (smooth) bounded domain in $\mathbb{R}^2$ or $\mathbb{R}^3$, and let u be some continuous function defined on $\overline{\Omega} = \Omega \cup \partial\Omega$. Suppose also that throughout Ω we have the condition:*

$$\Delta u = 0 , \qquad (u \text{ is harmonic in } \Omega) .$$

Then exactly one of the following must be true:

1. *$u \equiv const$ in $\overline{\Omega}$.*
2. *For each fixed $\vec{x} \in \Omega$ we have that:*

$$\min_{\partial\Omega} u < u(\vec{x}) < \max_{\partial\Omega} u .$$

Remark 7.2. The "strong" label on the above theorem refers to the fact that there are *strict* inequalities in case (2). It should be noted that these strict bounds are *not* uniform as one approaches the boundary. For example, if one approaches the boundary point $\vec{x}_0 \in \partial\Omega$ such that $\max_{\partial\Omega} u = u(\vec{x}_0)$ through a sequence of points $\vec{x}_n \in \Omega$, we are easily going to have that $\lim_{n\to\infty} u(\vec{x}_n) = u(\vec{x}_0)$. Thus, strict inequality becomes equality in the limit, so $u(\vec{x}_n)$ is becoming arbitrarily close to its max on the boundary (although it is strictly less for each *fixed* $\vec{x}_n$!).

Example 7.3. Here is a basic example of the above Theorem in action. Suppose that one is solving the specific problem in the unit disk $D_1 \subseteq \mathbb{R}^2$:

$$\begin{aligned} \Delta u &= 0 , \quad \text{when } r < 1 , \\ u|_{r=1} &= x^2 = \cos^2(\theta) . \end{aligned}$$

Then without computing anything, we automatically know that for each fixed $(x, y) \in D_1$ we have that $0 < u(x, y)$. That is, u is *strictly positive* in the interior of the disk $D_1 = \{(x, y)\big| x^2 + y^2 < 1\}$. This is because we clearly have $\min_{r=1} u = 0$, and we just need to apply Case (2) of the maximum principle above (clearly the variable boundary conditions do not allow for Case (1) of Theorem 7.1).

8. Uniqueness and Stability

It turns out that the problems (2b)–(2f) listed in Section 3 above have unique and "stable" solutions (as long as we specify $a > 0$ in (2f), which is physically reasonable cf. the discussion Section 4). By uniqueness we mean that there can only be *one* solution which corresponds to the given boundary data. However, a little care must be taken to define uniqueness in the case of the Neumann problem (2d). This is because we can always add a constant to any given solution to problem (2d) and produce another solution with the same boundary conditions. But in this case we just say there is uniqueness "modulo constants".

Stability is a somewhat less precise concept that we will not take the time to define rigorously (and there are many mathematically precise ways to measure how "stable" a PDE can be). But often these definitions can obscure the underlying physical "reality", which is simple the following: *if one makes small changes to the boundary data of the problems* (2b)–(2f)*, then the corresponding solutions u can only change by a small amount.* That is, it is impossible to produce a huge change in the solution u by making very small perturbations to the boundary data. This is physically reasonable because, for example, one should not be able to make some crazy change in a steady state temperature distribution of an oven by slightly changing the temperature of the walls of the oven. This would make adjusting the temperature of ovens a very tricky business!

As far as we are concerned, we can establish existence and uniqueness from two sources:

(1) Maximum principles.
(2) Energy estimates.

For example, the maximum principle in case (2) of Theorem 7.1 of the last section simply says that the solution u to the Dirichlet problem (2b) with $f = 0$ is forced to be sandwiched in between the extreme boundary values $\min_{\partial\Omega} u$ and $\max_{\partial\Omega} u$. This is obviously some form of stability. In general, we have the following result:

Theorem 8.1 (Uniqueness and Stability for Boundary Value Problems)**.** *The solutions to the problems* (2b)–(2f) *are unique in the following sense:*

(1) *For given* f, g *in problem* (2b)*, there can be only* one *solution* u*.*

(2) *For given* f, h *in problem* (2d)*, any two solutions* u, v *must satisfy* $u-v = c$*, where* c *is a constant. That is, every solution to* (2d) *can be written as some constant plus a* fixed *solution* u_0*. In particular,* ∇u *is the same for every solution to* (2d)*.*

(3) *If* $a > 0$ *in problem* (2f)*, then for a given* f, h *there is only* one *solution* u*.*

Finally, all of the solutions to the above problems are stable in the sense that small changes to the boundary data (g or h) produces small changes to the solutions (∇u in case of problem (2d)*).*

Remark 8.2. The proof of the first case, (1) above, follows easily from the maximum principle of Theorem 7.1. Just take two solutions u, v with the same boundary data and same f, and apply the maximum principle to $w = u - v$. You easily get that $w = 0$. As we have already remarked, stability is inherent in maximum principle type estimates.

The proofs of uniqueness and stability in cases (2) and (3) in this last Theorem are best seen from the point of view of *energy estimates*. See Section 13 below.

9. Average Value Properties

Because of the symmetry properties of the Laplace operator Δ, in particular the fact that it commutes with rotations (see Appendix A below), solutions of Δu possess remarkable "average value" properties with respect to integrations over rotationally symmetric sets. These average value properties are essentially the same in either $2D$ or $3D$, the only difference being that one has to compute correctly the volumes, areas, and lengths of balls, disks, spheres, and circles. Because of this, we'll state the average value properties for the cases of $2D$ and $3D$ separately. A bit of general terminology first: if Ω is some subdomain of $\mathbb{R}^2$ or $\mathbb{R}^3$, then we usually write:

$$AV_\Omega(u) \ = \ \frac{1}{|\Omega|} \int_\Omega u \ . \tag{5}$$

This is just the *average* of u over the set Ω. Here $|\Omega|$ is our notation for the volume or area of Ω. That is $|\Omega| = \int_\Omega 1$. There is an analogous notion for the integration over some closed subset of $\mathbb{R}^2$ or $\mathbb{R}^3$ that is not an open domain (like Ω). For instance, one may wish to look at averages over circles in $\mathbb{R}^2$, or spheres in $\mathbb{R}^3$. We won't bother to write down notation for all of this, but you should simply keep in mind that its looks a lot like (5) above, and that this notion of averages is implicit in all of the specific formulas that follow. The main average value formulas for harmonic functions are:

Theorem 9.1 (Average Value Formulas in $\mathbb{R}^2$)**.** *Denote by:*

$$D_r(x_0, y_0) = \{(x, y) \big| (x - x_0)^2 + (y - y_0)^2 < r^2\} \ ,$$

the disk of radius r centered at $(x_0, y_0) \in \mathbb{R}^2$. Suppose that $\Delta u = 0$ for all points inside of $D_r(x_0, y_0)$, then one has the following identities:

$$u(x_0, y_0) = \frac{1}{2\pi r} \int_{\partial D_r(x_0,y_0)} u \, d\ell \ , \tag{6}$$

$$u(x_0, y_0) = \frac{1}{\pi r^2} \iint_{D_r(x_0,y_0)} u \, dxdy \ . \tag{7}$$

Here $d\ell$ denotes arc length on the circle of radius r. That is, in polar coordinates centered at (x_0, y_0) it can be written as $d\ell = r d\theta$ (see Appendix A below).

Theorem 9.2 (Average Value Formulas in $\mathbb{R}^3$)**.** *Denote by:*

$$B_r(x_0, y_0, z_0) = \{(x, y, z) \big| (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 < r^2\} \ ,$$

the ball of radius r centered at $(x_0, y_0, z_0) \in \mathbb{R}^3$. Suppose that $\Delta u = 0$ for all points inside of $B_r(x_0, y_0, z_0)$, then one has the following identities:

$$u(x_0, y_0, z_0) = \frac{1}{4\pi r^2} \iint_{\partial B_r(x_0,y_0,z_0)} u \, dS \ , \tag{8}$$

$$u(x_0, y_0, z_0) = \frac{1}{\frac{4}{3}\pi r^3} \iiint_{B_r(x_0,y_0,z_0)} u \, dxdydz \ . \tag{9}$$

Here dS denotes surface area on the sphere of radius r. That is, in polar coordinates centered at (x_0, y_0, z_0) it can be written as $dS = r^2 \sin(\phi) d\phi d\theta$ (see Appendix A below).

Remark 9.3. As we have noted in class (see also p. 162 of the text), these average value identities are the key to proving the strong maximum principle Theorem 7.1 above.

Example 9.4. Here is an example involving an average value identity. Consider the harmonic function $u(x, y) = x^3 - 3xy^2$. Its easy to see that $u(0, 0) = 0$. Therefore, by equation (6) above, we have that:

$$0 = \frac{1}{2\pi r} \int_{\partial B_r} (x^3 - 3xy^2) d\ell \ ,$$

for any circle $\partial B_r = \{(x, y) \big| x^2 + y^2 = r^2\}$ centered at the origin in $\mathbb{R}^3$. Expanding this out in terms of polar coordinates, we immediately see that we must have:

$$0 = \int_0^{2\pi} \big(\cos^3(\theta) - 3\cos(\theta)\sin^2(\theta) \big) \, d\theta \ .$$

This identity can be verified by using some messy expansions involving double angle identities, but it is more enlightening to understand that the reason "why" this integral vanishes is that it represents the average boundary value of some harmonic function which vanishes at the origin!

10. Poisson's Formulas on $\mathbb{R}^2$

We now turn to a specific formula that allows one to compute exactly the solution to the problem:

$$\begin{aligned} \Delta u &= 0 , \quad \text{in } r < a , \\ u|_{r=a} &= h(\theta) . \end{aligned} \tag{10}$$

We have several ways to do this. The first is:

Theorem 10.1 (Poisson's Formula I)**.** *Suppose that u is a solution to the problem (10). Then we have the explicit formula for u inside the disk $r < a$:*

$$u(r,\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty} \left(\frac{r}{a}\right)^n (A_n \cos(n\theta) + B_n \sin(n\theta)) , \tag{11}$$

where the Fourier coefficients are computed from $h(\theta)$ explicitly as follows:

$$\begin{aligned} A_n &= \frac{1}{\pi}\int_0^{2\pi} h(\theta)\cos(n\theta)\ d\theta , \\ B_n &= \frac{1}{\pi}\int_0^{2\pi} h(\theta)\sin(n\theta)\ d\theta . \end{aligned}$$

By substituting the formulas for A_n and B_n back into the series (11), we can eliminate everything in favor of $h(\theta)$:

Theorem 10.2 (Poisson's Formula II)**.** *Suppose that u is a solution to the problem (10). Then we have the explicit formula for $r < a$:*

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_0^{2\pi} \frac{1}{a^2 - 2ar\cos(\theta - \phi) + r^2} h(\phi)\ d\phi . \tag{12}$$

To get a handle on this last formula, it is useful to compute it in terms of rectangular coordinates. This can be done by using the law of cosines. The resulting formula is:

Theorem 10.3 (Poisson's Formula III)**.** *Suppose that u is a solution to the problem (10). Then we have the explicit formula for $\sqrt{x^2+y^2} < a$:*

$$u(\vec{x}) = (a^2 - |\vec{x}|^2)\int_{r=a} \frac{1}{|\vec{x} - \vec{y}|^2} h(\vec{y})\ d\hat{\ell} . \tag{13}$$

Here we have set $(x,y) = \vec{x}$. Also, $d\hat{\ell} = \frac{1}{2\pi a}d\ell$ is the unit normalized arc length on the circle $r = 0$. In polar coordinates we simply have $d\hat{\ell} = \frac{d\theta}{2\pi}$.

Example 10.4. Lets return to the case of Example 7.3 above. Here we want to solve (10) with $h = \cos^2(\phi)$ on the boundary when $r = 1$. Using (12) we have the explicit formula:

$$u(r,\theta) = \frac{1 - r^2}{2\pi}\int_0^{2\pi} \frac{1}{1 - 2r\cos(\theta - \phi) + r^2} \cos^2(\phi)\ d\phi \tag{14}$$

From this, it is immediate that $0 < u(r,\theta)$ whenever $r < 1$, because all terms in the integral are easily seen to be positive. Positivity is perhaps even more clear using formula (13) above. This is a basic feature of the Dirichlet problem (with $f = 0$): *non-negative boundary values lead to* <u>*strictly*</u> *positive interior solutions unless the solution is identically zero.*

Physically this means that the heat at *any* point inside of a container, where the walls of the container are kept at zero temperature or above, must be strictly above zero unless you keep all of the walls perfectly at zero temperature. This is perhaps a bit counterintuitive at first, because one could be keeping the container walls at zero temperature except on a *very* small portion of the boundary. But even that small portion of the boundary is enough to ruin things *everywhere* inside the container (although perhaps only by a very little bit).

Example 10.5. This concerns the quite striking "smoothing effect" which happens when one solves the equation (10). Suppose that one solves this system when the boundary data h is bounded, but not necessarily even continuous. There was an example like this in the first HW. Then it turns out that the interior solution u is automatically smooth, just from the boundedness of h! To see this, we can write everything in rectangular coordinates using (13). Now, an easy calculation shows that:

$$\partial_x(|\vec{x}|^2) \ = 2x \ , \qquad\qquad \partial_x\left(\frac{1}{|\vec{x}-\vec{y}|^2}\right) \ = \ -\frac{2(x-\vec{y}^x)}{|\vec{x}-\vec{y}|^4} \ ,$$

where $\vec{y}$ is some other fixed vector in $\mathbb{R}^3$ (i.e. does not depend on $\vec{x}$). Also, we are using the notation where $\vec{y}^x$ denotes the x component of the vector $\vec{y}$. Using this and the product rule we see that:

$$\partial_x u(\vec{x}) \ = \ -2x\int_{r=a}\frac{1}{|\vec{x}-\vec{y}|^2}h(\vec{y})\ d\hat{\ell} \ - \ (a^2-|\vec{x}|^2)\int_{r=a}\frac{2(x-\vec{y}^x)}{|\vec{x}-\vec{y}|^4}h(\vec{y})\ d\hat{\ell} \ .$$

The upshot is that if $|\vec{x}| < a = |\vec{y}|$ the denominator of the integrals in the above formula can never be zero, so both integrands are bounded (also assuming that h is bounded). Thus, we see that $\partial_x u$ is actually bounded, and a similar argument clearly works if the derivative in question was $\partial_y u$. In fact, we can simply differentiate the formula (13) as many times as we want as long as we keep to the interior $|\vec{x}| < a = |\vec{y}|$. This shows that u is a very smooth function, even if its boundary data is not so smooth! Of course, the bounds on these high derivatives will start to be very nasty as soon as we let $|\vec{x}|$ get close to a, unless of we have that h is smooth with respect to θ (although this is not at all obvious from looking at (13)).

11. Complex Notation and Harmonic Polynomials (**not required for Exam 1**)

<u>MORE LATER</u>.

12. General Green's Identities

We record here two very useful identities. These are true in any dimension, for example on domains in $\mathbb{R}^2$ and $\mathbb{R}^3$. The importance of these formulas is that they allow us to derive other useful identities if we assume that the various functions involved solve some instance of Laplace's Equation. In other words, one can think of these formulas as being a sort of "black box" which can then produce other identities tailored to the given situation at hand.

Theorem 12.1 (Green's Identities)**.** *Let u and v be any two smooth functions defined in a bounded domain (smooth boundary) Ω of either $\mathbb{R}^2$ or $\mathbb{R}^3$. Then one has the following two identities:*

$$\int_{\partial\Omega} v\ \hat{n}\cdot\nabla u\ dS \ = \ \int_{\Omega} \nabla v\cdot\nabla u\ d\vec{x} + \int_{\Omega} v\Delta u\ d\vec{x}\ , \tag{15}$$

$$\int_{\partial\Omega} (v\ \hat{n}\cdot\nabla u - u\ \hat{n}\cdot\nabla v)\ dS \ = \ \int_{\Omega} (v\Delta u - u\Delta v)\ d\vec{x}\ . \tag{16}$$

Remark 12.2. The proof of (15) is not hard, and simply boils down to plugging the vector (in say the case of $\mathbb{R}^3$):

$$\vec{F} \ = \ v\nabla u \ = \ (vu_x, vu_y, vu_z)\ ,$$

into the Gauss-Green-Stokes Theorem 15.6 of Appendix B.

The proof of (16) follows by simply taking the difference of two versions of (15), where one interchanges the role of u and v in the second instance.

We also note here that once can rewrite these identities in terms of the inner product notation of subsection 2.1 above. For example, formula (15) becomes:

$$\big\langle v, \frac{\partial u}{\partial\hat{n}}\big\rangle_{\partial\Omega} \ = \ \langle\nabla v, \nabla u\rangle_{\Omega} + \langle v, \Delta u\rangle_{\Omega}\ .$$

Example 12.3. The way to use the above formulas (15)–(16) is to stick in various specific functions of interest. This is almost always done by choosing u to be the specific solution to the Laplace equation that you care about, and then letting v be something else like 1, or again equal to u, or perhaps some even more clever choice (see section 14 below). For example, suppose here that $\Delta u = f$ in Ω and also that $\hat{n}\cdot\nabla u = h$ on $\partial\Omega$. Then setting $v = 1$ in formula (15) we see that:

$$\int_{\partial\Omega} h\ dS \ = \ \int_{\Omega} f\ d\vec{x}\ .$$

This is precisely the solution to problem 11 on p. 154 of the text.

13. Energy Identities

Energy identities are integral formulas for solutions to the Laplace equation. They are often used to prove things like uniqueness and stability for various problems (e.g. Dirichlet, Neumann, or Robin). They are also very closely related to what is known as "Dirichlet's Principle", which says that harmonic functions are energy minimizers (see Section 15 below).

The basic energy estimate for the Laplace equation is the following, which can easily be derived from (15) by setting $v = u$:

Theorem 13.1 (The Energy Estimate)**.** *Suppose that u is some function defined on a bounded domain Ω inside of either $\mathbb{R}^2$ or $\mathbb{R}^3$, and that furthermore we have that $\Delta u = 0$. Then we also have:*

$$\int_{\partial\Omega} u\ \hat{n}\cdot\nabla u\ dS \ = \ \int_{\Omega} |\nabla u|^2\ d\vec{x}\ . \tag{17}$$

We refer to:

$$E_{\Omega}[u] \ = \ \frac{1}{2}\int_{\Omega} |\nabla u|^2\ d\vec{x}\ , \tag{18}$$

as the "energy" of u in Ω.

Example 13.2. Lets see how the identity (17) leads almost immediately to uniqueness for certain boundary value problems. Suppose that u and v are both solutions to the Neumann type problem (2d). Then the difference $w = u - v$ solves the following boundary value problem:

$$\begin{aligned} \Delta w \ &= \ 0\ , \quad \text{in } \Omega\ , \\ \hat{n}\cdot\nabla w|_{\partial\Omega} \ &= \ 0\ . \end{aligned} \tag{19}$$

Plugging this information into (17) above we see that:

$$\int_{\Omega} |\nabla w|^2\ d\vec{x} \ = \ 0\ .$$

They key now is that the quantity $|\nabla w|^2$ in this integral is always non-negative. Thus, because its integral is zero it must in fact be zero at each point of Ω. That is $|\nabla w|^2 \equiv 0$ in Ω. But if a function's gradient vanishes inside of some (connected) domain, then that function is a constant in that domain. Thus $w = const$ in Ω. Therefore we must have that:

$$v \ = \ u + c\ ,$$

for some appropriate constant c. This is exactly what we stated in Case (2) of Theorem 8.1 above.

14. Representation Formulas and Fundamental Solutions

We now state some explicit, but very general, representation formulas for solutions to the Laplace equation in $\mathbb{R}^2$ and $\mathbb{R}^3$. The goal is to be able to write a general solution to $\Delta u = 0$ at points inside of a domain Ω in terms of its values on $\partial\Omega$. To to this, we will use a specific instance of the second Green's identity (16) above. For the function u we simply take our solution to $\Delta u = 0$. The function v

however is not as obvious. What we will do is to take v to be the following dimension dependent functions, which are often referred to as *fundamental solutions* to the Laplace equation:

$$v_{\vec{x_0}}(\vec{x}) = \frac{1}{2\pi}\ln(|\vec{x}-\vec{x}_0|) , \qquad \text{in } \mathbb{R}^2 , \tag{20}$$

$$v_{\vec{x_0}}(\vec{x}) = \frac{-1}{4\pi|\vec{x}-\vec{x}_0|} , \qquad \text{in } \mathbb{R}^3 . \tag{21}$$

The reason that this is useful stems from the following calculation:

Theorem 14.1 (Fundamental Solution Identities)**.** *Let $\Delta u = 0$ be a solution to Laplace's equation inside of a bounded domain Ω contained in either $\mathbb{R}^2$ or $\mathbb{R}^3$. Also, in the case of $\mathbb{R}^2$ let $D_\epsilon(\vec{x_0})$ be a small disk of radius ϵ contained in Ω. In the case of $\mathbb{R}^3$ let $B_\epsilon(\vec{x_0})$ be a small ball of radius ϵ contained in Ω. Then one has the following identities:*

$$\frac{1}{2\pi\epsilon}\int_{\partial D_\epsilon(\vec{x}_0)} (u - \epsilon\ln\epsilon\,\frac{\partial u}{\partial\hat{n}})\ d\ell = \int_{\partial\Omega}\left[u(\vec{x})\frac{\partial}{\partial\hat{n}}v_{\vec{x_0}}(\vec{x}) - \frac{\partial u}{\partial\hat{n}}(\vec{x})\,v_{\vec{x_0}}(\vec{x})\right]\ d\ell(\vec{x}) , \tag{22}$$

$$\frac{1}{4\pi\epsilon^2}\iint_{\partial B_\epsilon(\vec{x}_0)} (u - \epsilon\,\frac{\partial u}{\partial\hat{n}})\ dS = \iint_{\partial\Omega}\left[u(\vec{x})\frac{\partial}{\partial\hat{n}}v_{\vec{x_0}}(\vec{x}) - \frac{\partial u}{\partial\hat{n}}(\vec{x})\,v_{\vec{x_0}}(\vec{x})\right]\ dS(\vec{x}) . \tag{23}$$

Here again $d\ell$ and dS denote arclength and surface area integrals. On the right hand side we use the notation $d\ell(\vec{x})$ and $dS(\vec{x})$ to indicate that the integration is with respect to the $\vec{x}$ variable (i.e. we are holding $\vec{x}_0$ equal to constant vector). Here, as always, $\hat{n}$ denotes the outward unit normal to $\partial D_\epsilon(\vec{x}_0)$, $\partial B_\epsilon(\vec{x}_0)$, or $\partial\Omega$, depending on the context.

Remark 14.2. The proof of both of these identities follows simply from calculations we have already done. In fact, all one has to do is to simply plug the functions $v_{\vec{x_0}}(\vec{x})$ and u into the second Green's identity (16), and then use the (translated) versions of equations (3)–(4) from Theorem 6.1 of Section 6 above. For example, in the case of $\Omega \subseteq \mathbb{R}^3$ we have from translating equation (4) with $B = 0$ that:

$$\Delta\left(\frac{1}{4\pi|\vec{x}-\vec{x}_0|}\right) = 0 ,$$

as long as we are in the region $\widetilde{\Omega} = \Omega \cap [B_\epsilon(\vec{x}_0)]^{\mathcal{C}}$, where we use $[B_\epsilon(\vec{x}_0)]^{\mathcal{C}}$ to denote the *exterior* of the ball $B_\epsilon(\vec{x}_0)$. That is, $\widetilde{\Omega}$ is simply the original domain Ω with a small ball centered at $\vec{x}_0$ cut out of it, and the function $v_{\vec{x_0}}(\vec{x})$ is smooth there as long as $0 < \epsilon$ because we stay away from the singularity.

Next, notice that we have $\partial\widetilde{\Omega} = \partial\Omega \cup \partial B_\epsilon(\vec{x}_0)$, so by applying the Green's identity (16) to the functions $v_{\vec{x}_0}(\vec{x})$ and u (recall that also $\Delta u = 0$) we easily have that:

$$(24) \quad 0 \ = \ \iint_{\partial B_\epsilon(\vec{x}_0)} \Big[u\partial_{\hat{n}}\big(\frac{-1}{4\pi|\vec{x}-\vec{x}_0|}\big) - \frac{-1}{4\pi|\vec{x}-\vec{x}_0|}\frac{\partial u}{\partial \hat{n}} \Big] \ dS \\ + \ \frac{1}{4\pi}\iint_{\partial\Omega} \left[u(\vec{x})\frac{\partial}{\partial \hat{n}} v_{\vec{x}_0}(\vec{x}) - \frac{\partial u}{\partial \hat{n}}(\vec{x})\, v_{\vec{x}_0}(\vec{x}) \right] \ d\ell(\vec{x}) \ .$$

The important thing to keep in mind when doing these calculations is that since $\widetilde{\Omega}$ is in the exterior of $B_\epsilon(\vec{x}_0)$, the unit normals in the first main term on the right hand side above are pointing inward. For example, if $\vec{x}_0 = (0,0,0)$ is the origin in $\mathbb{R}^3$, then we simply have $\partial_{\hat{n}} = -\partial_r$. In any case, one can always compute these derivatives by translating back to the origin, so in particular we have that:

$$\frac{-1}{4\pi|\vec{x}-\vec{x}_0|}\Big|_{\partial B_\epsilon(\vec{x}_0)} \ = \ -\frac{1}{4\pi\epsilon} \ , \\ \partial_{\hat{n}}\big(\frac{-1}{4\pi|\vec{x}-\vec{x}_0|}\big)\Big|_{\partial B_\epsilon(\vec{x}_0)} \ = \ -\frac{1}{4\pi\epsilon^2} \ .$$

By simply substituting these last two lines into the first term on the right hand side of (24) above, and then collecting the resulting expression onto the left hand side of the equation, we arrive at (23).

A similar calculation which we leave to the HW works to prove (23).

Next, one should notice that when the terms involving the extra ϵ's and the derivative $\frac{\partial u}{\partial \hat{n}}$ on the right hand side of (22) and (23) are integrated, they can actually equal be evaluated and they are always zero. This follows easily from the fact that $\Delta u = 0$ and the first Green's identity (15) with $v = 1$ applied to the interior of $D_\epsilon(\vec{x}_0)$ or $B_\epsilon(\vec{x}_0)$ (whichever is the case). However, this observation is not entirely necessary because in any case we will take the limit of the left hand side of (22)–(23) as $\epsilon \to 0$. Computing this in polar or spherical coordinates, the reader will easily verify that in either case we have:

$$\lim_{\epsilon\to 0} \ \frac{1}{2\pi\epsilon}\int_{\partial D_\epsilon(\vec{x}_0)} (u - \epsilon\ln\epsilon\,\frac{\partial u}{\partial \hat{n}}) \ d\ell \ = \ u(\vec{x}_0) \ , \qquad \text{on } \mathbb{R}^2 \ , \\ \lim_{\epsilon\to 0} \ \frac{1}{4\pi\epsilon^2}\iint_{\partial B_\epsilon(\vec{x}_0)} (u - \epsilon\,\frac{\partial u}{\partial \hat{n}}) \ dS \ = \ u(\vec{x}_0) \ , \qquad \text{on } \mathbb{R}^3 \ .$$

We may now substitute these formulas into the left hand side of equations (22)–(23) because the right hand side *does not depend on* ϵ. Doing this, we arrive at the following fundamental formulas:

Theorem 14.3 (Representation Formulas for Harmonic Functions)**.** *Let u be a function defined on a bounded domain Ω in either $\mathbb{R}^2$ or $\mathbb{R}^3$, and suppose that $\Delta u = 0$. Suppose also that $\vec{x}_0 \in \Omega$ is any fixed point.*

With this setup, in the case of $\mathbb{R}^2$ we have that:

$$(25) \qquad u(\vec{x}_0) \ = \ \frac{1}{2\pi}\int_{\partial\Omega} \left[u(\vec{x})\frac{\partial}{\partial \hat{n}}\big(\ln(|\vec{x}-\vec{x}_0|)\big) - \frac{\partial u}{\partial \hat{n}}(\vec{x})\,\ln(|\vec{x}-\vec{x}_0|) \right] \ d\ell(\vec{x}) \ .$$

Likewise, in the case of $\mathbb{R}^3$ *we have that:*

$$u(\vec{x}_0) \ = \ \frac{1}{4\pi}\iint_{\partial\Omega}\Big[-u(\vec{x})\frac{\partial}{\partial\hat{n}}\big(\frac{1}{|\vec{x}-\vec{x_0}|}\big)+\frac{\partial u}{\partial\hat{n}}(\vec{x})\,\frac{1}{|\vec{x}-\vec{x_0}|}\Big]\ dS(\vec{x})\ . \tag{26}$$

Remark 14.4. Notice that the integrals express the value $u(\vec{x}_0)$ purely in terms of an explicit integral of the boundary data. However, notice that the price one pays for this is that we have to know *both* Dirichlet and Neumann type boundary conditions, which is not so natural. From our discussion of uniqueness, it is clear that one cannot arbitrarily specify both Dirichlet and Neumann boundary conditions, and there must always be some nontrivial relation between the two. Therefore, the explicit formulas (25)–(26) have some kind of subtle redundancy inherent in them. We will return shortly to the problem of eliminating this redundancy.

15. The Dirichlet Principle (**not required for Exam 1**)

In this section we will discuss a certain mathematical property of solutions to $\Delta u = 0$ that in some sense goes to the heart of their "physical behavior". This is not a specific formula for solutions, but rather a qualitative description somewhat akin to the maximum principle of Section 7. However, unlike the maximum principle, this property is much more specific in that it completely characterizes solutions to the (homogeneous) Laplace equation.

To get things started, we recall the notion of the "potential energy" of a function u defined in some bounded domain Ω, which can be taken in either $\mathbb{R}^2$ or $\mathbb{R}^3$. The energy is (see also line (18)):

$$E[u] \ = \ \frac{1}{2}\int_\Omega |\nabla u|^2\ d\vec{x}\ . \tag{27}$$

Quantities like this are already familiar to us from our discussion in Section 13. There usefulness stems largely from the following fundamental characterization of harmonic functions:

Theorem 15.1 (The Dirichlet Principle)**.** *Let u be a function on either $\mathbb{R}^2$ or $\mathbb{R}^3$, and Ω a bounded set with smooth boundary. Furthermore, suppose that u is defined according to the following problem:*

$$\Delta u \ = \ 0\ , \quad in\ \Omega\ , \tag{28}$$

$$u|_{\partial\Omega} \ = \ g\ . \tag{29}$$

Then u is the (unique) function which minimizes the energy E among all competing functions with the boundary value g. That is, if v is any *other function such that $v|_{\partial\Omega} = g$, then one must have:*

$$E[u] \ \leqslant \ E[v]\ . \tag{30}$$

Remark 15.2. The proof of the bound (30) is not hard, and simply boils down to an application of the first Green's identity (15). The idea is just to write:

$$|\nabla v|^2 \ = \ |\nabla u|^2 + |\nabla(v-u)|^2 + 2\nabla(u)\cdot\nabla(v-u) \ ,$$

and to apply the Green's identity to the last term after you integrate it.

Remark 15.3. For those familiar with physics, you can think of the Dirichlet principle as being a static version of the famous "principle of least action" for mechanical systems. In fact, we will make this intuition more precise when we talk about elastic and acoustic waves. It is a basic principle that physical systems, if left in isolation, prefer to assume the configuration of "least energy". For example, according to our discussion in Section 4, we may interpret the solution u to the problem (29) as the steady-state temperature distribution inside a body with boundary temperature distribution equal to g. In this case, the Dirichlet principle says that this steady state is formed when the temperature u in the body is spread out in such a way as to minimize its "potential energy", defined by (27).

Appendix A: Polar and Spherical Coordinates. Because of the special relationship between the Laplace equation and rotations in the plane (i.e. $\mathbb{R}^2$) and in space (i.e. $\mathbb{R}^3$), it is often times convenient to use polar or spherical coordinates in place of the usual rectangular coordinates when doing computations. Recall that every point $(x,y)\in\mathbb{R}^2$ can be expressed in the form:

$$\begin{aligned} x \ &= \ r\cos(\theta) \ , \\ y \ &= \ r\sin(\theta) \ . \end{aligned}$$

Here we enforce the restrictions $0\leqslant r<\infty$ and $0\leqslant\theta<2\pi$, so that every point other than $(0,0)$ has a unique representation. Of course $r=0$ corresponds to the point $(0,0)$ for *any* value of θ. With respect to these coordinates we may write the $2D$ Laplace equation as:

$$\Delta \ = \ \partial_r^2 + \frac{1}{r}\partial_r + \frac{1}{r^2}\partial_\theta^2 \ .$$

The other important identities that we will use many times are the formulas for line and area elements in these coordinates. Specifically, we have that:

$$\begin{aligned} dxdy \ &= \ rdrd\theta \ , \\ d\ell \ &= \ rd\theta \ , \end{aligned}$$

where $d\ell$ denotes the arclength on the circle of radius r centered at the origin (this can easily be translated to circles centered at other points).

Example 15.4. To see an example of the above formulas, suppose that we wish to integrate the expression Δu over the annulus $\epsilon<r<a$ centered at the origin.

Then in polar coordinates this integral becomes:

$$\begin{aligned}
\iint_{\epsilon<r<a} \Delta u \ dxdy &= \int_{\epsilon}^{a}\int_{0}^{2\pi} \left(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} \right)(r,\theta)\ rdrd\theta \ , \\
&= \int_{\epsilon}^{a}\int_{0}^{2\pi} \left(u_{rr} + \frac{1}{r}u_r \right)(r,\theta)\ rdrd\theta + \int_{\epsilon}^{a} \frac{dr}{r}\big[u_\theta(r)\big]\big|_{\theta=0}^{\theta=2\pi} \ , \\
&= \int_{\epsilon}^{a}\int_{0}^{2\pi} \left(u_{rr} + \frac{1}{r}u_r \right)(r,\theta)\ rdrd\theta \ , \\
&= \int_{0}^{2\pi} u_r(a,\theta) a d\theta - \int_{0}^{2\pi} u_r(\epsilon,\theta)\epsilon d\theta \ .
\end{aligned}$$

Allowing $\epsilon \to 0$ in this last expression, and reverting to arclength notation, we have that for any function u:

$$\iint_{r<a} \Delta u \ dxdy \ = \ \int_{r=a} u_r(a,\theta) d\ell \ .$$

In particular, if $\Delta u = 0$, we have the rule that:

$$\partial_r \int_0^{2\pi} u(r,\theta)\ d\theta \ = \ 0,$$

which can also be written in the form:

$$\partial_r \left(\frac{1}{2\pi a} \int_{r=a} u(r,\theta)\ d\ell \right) \ = \ 0.$$

From this last line it is easy to deduce the mean value formula (6) for $2D$ harmonic functions.

The discussion of spherical coordinates on $\mathbb{R}^3$ is very similar. In this case we have the following formulas:

$$\begin{aligned}
x &= r\sin(\phi)\cos(\theta) \ , \\
y &= r\sin(\phi)\sin(\theta) \ , \\
z &= r\cos(\phi) \ .
\end{aligned}$$

To define almost all points uniquely, recall that we need to enforce:

$$\begin{aligned}
0 &\leqslant r < \infty \ , \\
0 &\leqslant \phi \leqslant \pi \ , \\
0 &\leqslant \theta < 2\pi \ .
\end{aligned}$$

In the case of spherical coordinates, a somewhat painful calculation shows that:

$$\Delta \ = \ \partial_r^2 + \frac{2}{r}\partial_r + \frac{1}{r^2}\left[\frac{1}{\sin^2(\phi)}\partial_\theta^2 + \partial_\phi^2 + \frac{\cos(\phi)}{\sin(\phi)}\partial_\phi \right] \ .$$

Also, in these coordinates we have the following useful identities for volume and area elements:

$$\begin{aligned}
dxdydz &= \sin(\phi) r^2 drd\phi d\theta \ , \\
dS &= \sin(\phi) r^2 d\phi d\theta \ .
\end{aligned}$$

Here dS denotes the surface area element on the sphere of radius r centered at the origin (it is easy to translate this formula to other spheres by simply using spherical coordinates based at their centers).

Example 15.5. Suppose that $u(x,y,z)$ is a function given in terms of rectangular coordinates. Then lets write down an expression for its average over the sphere of radius a centered at the origin. In this case we have that:

$$AV_{r=a}(u) \ = \ \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} u\big(a\sin(\phi)\cos(\theta), a\sin(\phi)\sin(\theta), a\cos(\phi)\big) \ \sin(\phi)d\phi d\theta \ .$$

Notice that this formula makes sense, because if we set $u \equiv 1$ then we get that the average is $AV_{r=a}(u) = 1$.

Appendix B: Some Formulas from Calc. III. We recall here two of the most basic facts from multivariable calculus. The first is:

Theorem 15.6 (The Gauss-Green-Stokes Identity)**.** *Suppose that $\vec{F}$ is a vector-field in a bounded domain Ω inside of either $\mathbb{R}^2$ or $\mathbb{R}^3$. Then we have that:*

$$\int_\Omega \nabla\cdot\vec{F}\ d\vec{x} \ = \ \int_{\partial\Omega} \hat{n}\cdot\vec{F}\ dS \ .$$

Here $\hat{n}$ denotes the outward unit normal to the domain Ω, and dS is the surface length or area on the boundary $\partial\Omega$.

The second is:

Theorem 15.7 (The Poincaré Lemma)**.** *Suppose that $\vec{F}$ is some irrotational vector-field in $\mathbb{R}^3$. That is, we have that $\nabla\times\vec{F} = 0$ where:*

$$\nabla\times\vec{F} \ = \ \Big(\partial_y\vec{F}^z - \partial_z\vec{F}^y, \partial_z\vec{F}^x - \partial_x\vec{F}^z, \partial_x\vec{F}^y - \partial_y\vec{F}^x\Big) \ .$$

Then there exists a scalar function $\phi(x,y,z)$ such that:

$$\vec{F} \ = \ \nabla\phi \ .$$

Example 15.8. Here is an example of the second result above. Suppose that we are given:

$$\vec{F} \ = \ (y,x,0) \ .$$

Then a direct computation verifies that $\nabla\times\vec{F} = 0$. Therefore we know that $\vec{F} \ = \ \nabla\phi$ for some ϕ. In fact, its not hard to see that $\phi(x,y,z) = xy$ works.

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