Homework for Math 132A
Partial Differential Equations
Spring 2009

We will use the following convention for listing exercises numbers: The *italicized* numbers refer to exercises for which there is a solution in the student manual. Therefore, these are just to be *studied*, but not turned in. Please make sure you understand this material for the exams. The **bold faced** numbers are to be solved and turned in. If you have any questions on this material, please ask the instructor or the TA during office hours or recitation.

Due Friday April 10:

- Section 6.1: **5**, **10**.
- Section 7.1: **1**, **3**, *7*, **9**.
- Section 7.2: *1*, *2*, **3**.

Now Due Friday April 24:

- Section 7.3: *1*.
- Section 7.4: **5**, *7*, **8**, **12**, **14**, *15*, **16**, **18**, *21*, **23**.

Due Friday May 6:

- Section 3.2: **1**, *2*.
- Section 3.2: Let $u(t,x)$ be a solution to the wave equation with one Dirichlet boundary condition:

$$\begin{aligned} \partial_t^2 u &= \partial_x^2 u \,, \qquad 0 \leqslant x \,, \\ u(t,0) &= 0 \,. \end{aligned}$$

Show that this function satisfies the conservation of energy, $E(t) \equiv E_0$, where:

$$E(t) \; = \; \frac{1}{2}\int_0^\infty (u_t^2 + u_x^2)(t,x)dx \; .$$

- Section 9.1: **3**, *5*, **6**, *7*.
- Section 9.2: **10**, *11*, **15**, **18**, *19*.

Due Friday May 22:

- Section 9.2: Find a closed form explicit solution to the 3D wave equation:

$$\begin{aligned} \partial_t^2 u \; &= \; \Delta_{\mathbb{R}^3} u \; , \\ u(0,\vec{x}) \; &= \; 0 \; , \\ u_t(0,\vec{x}) \; &= \; \begin{cases} 1 - |\vec{x}|^2 \; , & |\vec{x}| \leqslant 1; \\ 0 \; , & |\vec{x}| > 1. \end{cases} \end{aligned}$$

for all times $t \geqslant 2$. (Hint: This problem is spherically symmetric, so it should be of the form $u = \frac{F(r+t)+G(r-t)}{r}$).

- Section 9.2: Let u be a solution to the 2D wave equation:

$$\begin{aligned} \partial_t^2 u \; &= \; \Delta_{\mathbb{R}^2} u \; , \\ u(0,\vec{x}) \; &= \; 0 \; , \\ u_t(0,\vec{x}) \; &= g(\vec{x}) \; , \end{aligned}$$

and assume that $g(\vec{x})$ is bounded, rapidly decaying (e.g. $|g(\vec{x})| < (1+|\vec{x}|)^{100}$), and also such that:

$$\int_{\mathbb{R}^2} g(\vec{x})d\vec{x} \; = \; 0 \; .$$

Show that in this case, the solution $u(t,\vec{x})$ is such that:

$$|u(t,\vec{x})| \; \leqslant \; Ct^{-2} \; ,$$

for *fixed* $\vec{x}$, and $t \to \infty$. You should compare this with Exercise 18 in the previous HW.

- Section 9.2: (Hard) Let u be a solution to the 2D wave equation:

$$\begin{aligned} \partial_t^2 u \; &= \; \Delta_{\mathbb{R}^2} u \; , \\ u(0,\vec{x}) \; &= \; 0 \; , \\ u_t(0,\vec{x}) \; &= g(\vec{x}) \; , \end{aligned}$$

and assume that $g(\vec{x})$ is bounded, rapidly decaying (e.g. $|g(\vec{x})| < (1+|\vec{x}|)^{100}$), and also such that:

$$\int_{\mathbb{R}^2} g(\vec{x}) d\vec{x} \ = \ 0 \ .$$

Can you find an additional condition so that:

$$|u(t,\vec{x})| \ \leqslant \ Ct^{-4} \ ,$$

for *fixed* x, and $t \to \infty$?

- Section 9.3 (our version): Here is another description of a wave with very low angular dispersion. Consider the function:

$$u(t,x,y,z) \ = \ \big(\sqrt{\lambda^2-1}t-\lambda x)^2+y^2+z^2\big)^{-\frac{1}{2}}\bigg(\varphi\Big(\lambda t-\sqrt{\lambda^2-1}x+\sqrt{(\sqrt{\lambda^2-1}t-\lambda x)^2+y^2+z^2}\Big) - \varphi\Big(\lambda t-\sqrt{\lambda^2-1}x-\sqrt{(\sqrt{\lambda^2-1}t-\lambda x)^2+y^2+z^2}\Big)\bigg) \ ,$$

where again we are setting $\varphi(x) = e^{-|x|^2}$, and here $\lambda \geqslant 1$. Explain the following:

(1) Show that $u(t,x,y,z)$ solves the 3D wave equation. This may seem very hard at first, but here are some observations: a) The fact its a solution does not depend on the form of φ (it works for any function). b) If $\lambda = 1$ this is just a spherical wave. c) If $w(t,x,y,z)$ solves the 3D wave equation, then so does $w\big(\lambda t - \sqrt{\lambda^2-1}x, \sqrt{\lambda^2-1}t - \lambda x, y, z\big)$.

(2) Compute that for $|t| \leqslant \lambda$ one has the localization property:

$$|u(t,x,y,z)| \ \leqslant \ \frac{C}{1+\lambda|t-x|+|y|+|z|} \ ,$$

where C is an appropriate constant, and also that:

$$|u(t,x,y,z)| \ \sim \ 1 \ , \qquad \text{when} \ \ |t-x| < \lambda^{-1} \ , \ \ |y|+|z| \leqslant 1 \ .$$

This means that for $|t| \leqslant \lambda$, u is small away from a rectangle of size $\sim \lambda \times \lambda^{-1} \times 1 \times 1$ oriented along the light-ray $(t,t,0,0)$. Try to draw a picture of this. This wave maintains the much smaller dispersion in has the direction it is traveling (i.e. along x-axis). Waves of this form are sometimes called "Knapp counterexamples", and are used to show that some types of decay estimates for the wave equation are false.

- Section 9.4 (our version): In this problem we consider the heat equation with highly oscillatory initial data. Use the integral formula to write down an explicit integral for the solution of the problem:

$$\partial_t u \ = \ \Delta_{\mathbb{R}^3} u \ ,$$
$$u(0,x) \ = \ e^{i\vec{x}\cdot\xi_0}e^{-|\vec{x}|^2} \ .$$

Recall that the solution to the *wave equation* with this type of data forms a wave-packet that travels in the $\hat{\xi}_0$ direction for long time. Show that at time $t = 1$ one has in this case the decay:

$$|u(1, \vec{x})| \leqslant \frac{C_k}{(1 + |\xi_0| + |\vec{x}|)^k} ,$$

for any $k \geqslant 0$. That is, for this equation the higher the frequency of the initial data, the faster the solution becomes decoherent! (Hint: Do k integration by parts in the integral formula).