Practice Set I
Math 132

Problem 1: Consider the function on $\mathbb{R}^3$ given by $u(x,y) = e^{x+y+z}$. Show that this function obeys the strong form of the maximum principle on *any* bounded domain $\Omega \subseteq \mathbb{R}^3$, but that it is not harmonic.

Problem 2: Is there a function $u(r,\phi,\theta)$ (in 3D spherical coordinates) which solves the Neumann type problem:

$$\begin{aligned} \Delta u &= 0 \ , \quad \text{in } r < 1 \ , \\ \partial_r u|_{r=1} &= 1 + \cos(\theta) \ . \end{aligned}$$

If so, write down a solution. If not, say why.

Problem 3: Let $G(\vec{x},\vec{y})$ be the Green's function for a bounded domain $\Omega \subset \mathbb{R}^3$. Show that $G(\vec{x},\vec{y}) < 0$ for all $\vec{x},\vec{y} \in \Omega$. That is, the Green's function of any domain is always a strictly negative function away from the boundary.

Problem 4: Consider the function:

$$w(x,y,z) = \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}} \ .$$

Notice that this is the conformal inversion of the harmonic function $u(x,y,z) = z$ as discussed in class. Therefore, w is harmonic in $\mathbb{R}^3 \setminus (0,0,0)$. Show that the integral of w over the unit sphere vanishes, even though $w(0,0,0)$ is not defined (it basically can be any value depending on how one approaches the origin in $\mathbb{R}^3$). That is, show by direct calculation that:

$$\iint_{|\vec{x}|=1} w(\vec{x}) dS_{\vec{x}} = 0 \ .$$

Conclude from this that w *does not* obey the average value property for spheres $S_{r_0}(\vec{x}_0)$ that contain the origin $(0,0,0)$ in their interior even if $\vec{x}_0 \neq (0,0,0)$ (e.g. just notice that the value of the integral is continuous as one shifts the center of the sphere a little bit).

Problem 5: Consider two solutions u and v to the following Dirichlet problems:

$$\begin{aligned} \Delta u &= 0 \ , & \text{in} \quad |\vec{x}| < 1 \ , \\ u|_{|\vec{x}|=1} &= x^2 - y^2 \ , \end{aligned}$$

and:

$$\begin{aligned} \Delta v &= 0 , && \text{in} \quad |\vec{x}| < 1 , \\ v|_{|\vec{x}|=1} &= z^2 + 1 . \end{aligned}$$

Show that one must have $u(\vec{x}) < v(\vec{x})$ for every point $|\vec{x}| < 1$.

Problem 6: Consider the following harmonic function on $\mathbb{R}^3$, $u(x, y, z) = x^2 - y^2 + 2yz$. Compute the average of u over the unit sphere centered at the point $(1, 2, 0)$.

Problem 7: Let u and v be the solutions to the following $3D$ boundary value problems (i.e. in the unit ball $B_1 \subseteq \mathbb{R}^3$):

$$\begin{aligned} \Delta u &= 0 , \quad \text{in } r < 1 , \\ u|_{r=1} &= \sin(\theta) , \end{aligned}$$

and:

$$\begin{aligned} \Delta v &= 0 , \quad \text{in } r < 1 , \\ \partial_r v|_{r=1} &= \cos(\theta)\cos(2\phi) . \end{aligned}$$

Note that the first condition is Dirichlet, while the second is Neumann. Also, things have been written in terms of angular coordinates (θ, ϕ) with the convention that ϕ is the angle with the z axis, and θ is the usual polar angle in the plane $z = 0$. Notice that these can be taken as coordinates on the unit sphere ∂B_1.

Use the first Green's identity to show that u and v are orthogonal in the sense that:

$$\iiint_{B_1} \nabla v \cdot \nabla u \; d\vec{x} \; = \; 0 .$$