Practice Set II
Math 132

This section concerns the wave equation.

Problem 1: Consider the solution $u(t,x)$ to the following half-space problem:

$$\partial_t^2 u - \partial_x^2 u = 0 , \qquad \text{On } \mathbb{R}\times\mathbb{R}^+ ,$$

$$u(0,x) = \begin{cases} \sin(\frac{\pi}{2}x) , & 8 \leqslant x \leqslant 10 ; \\ 0 , & \text{otherwise} , \end{cases}$$
$$u_t(0,x) = 0 ,$$

$$u(t,0) = 0 .$$

Use the explicit formula derived in class to compute the value of the wave at the following space-time points:

$$(x_1,t_1) = (1,7) ,$$
$$(x_2,t_2) = (1,8) ,$$
$$(x_3,t_3) = (1,10) .$$

In particular, notice that this shows solutions to the wave equation can violate the maximum principle (e.g. in the form stated on p. 41 of the text for the heat equation, which also depends on time).

Problem 2: Consider the solution u to the two (space) dimensional wave equation:

$$\partial_t^2 u - c^2\Delta u = 0 , \qquad \text{on } \mathbb{R}\times\mathbb{R}^2 ,$$
$$u(0,\vec{x}) = f(\vec{x}) ,$$
$$u_t(0,\vec{x}) = g(\vec{x}) .$$

Suppose also it is the case that the initial data f and g vanish as follows:

$$f(\vec{x}) = g(\vec{x}) = 0 , \qquad |\vec{x}| > 1 ,$$

Furthermore, suppose you know the initial *total* energy $E(u)(0)$ is such that:

$$E(u)(0) \ = \ \frac{1}{2}\iint_{\mathbb{R}^2} (g^2 + |\nabla f|^2) \ d\vec{x} \ = \ 1 \ .$$

Compute the *local* energy inside the disk of radius $a = 1 + c$ at time $t = 1$. That is, compute the quantity:

$$E_{D_a}(u)(1) \ = \ \frac{1}{2}\iint_{x^2+y^2\leqslant(1+c)^2} (u_t^2 + |\nabla u|^2)(1, \vec{x}) \ d\vec{x} \ .$$

Problem 3: Consider the solution u to the two dimensional wave equation:

$$\begin{aligned} \partial_t^2 u - \Delta u \ &= \ 0 \ , \\ u(0, \vec{x}) \ &= \ 0 \ , \\ u_t(0, \vec{x}) \ &= \ e^{-|\vec{x}|^2} \ . \end{aligned}$$

Please answer the following:

- Show that the local energy in the unit disk centered at the origin at $t = 1$ is less than $\frac{\pi}{4}(1 - e^{-8})$. That is, prove the bound:

  $$\frac{1}{2}\iint_{x^2+y^2\leqslant 1} \big(u_t^2 + |\nabla u|^2\big)(1, x, y) \ dxdy \ \leqslant \ \frac{\pi}{4}(1 - e^{-8}) \ .$$

  (Hint: It is not hard to integrate a 2D Gaussian, i.e. $e^{-|\vec{x}|^2}$, using polar coordinates.)

- Conclude from this bound and the conservation of total energy that one also has the following *lower* bound on the energy *outside* the unit disk at time $t = 1$:

  $$\frac{\pi}{4}e^{-8} \ \leqslant \ \frac{1}{2}\iint_{x^2+y^2>1} \big(u_t^2 + |\nabla u|^2\big)(1, x, y) \ dxdy \ .$$

Problem 4: Consider the solution u to the three (space) dimensional wave equation:

$$\begin{aligned} \partial_t^2 u - c^2\Delta u \ &= \ 0 \ , \qquad\qquad \text{On } \mathbb{R}\times\mathbb{R}^3 \ , \\ u(0, \vec{x}) \ &= \ f(\vec{x}) \ , \\ u_t(0, \vec{x}) \ &= \ g(\vec{x}) \ . \end{aligned}$$

Suppose one has the following value for the local energy at time $t = 5$:

$$E_{B_1}(u)(5) \ = \ \frac{1}{2}\iint_{x^2+y^2+z^2\leqslant 1} (u_t^2 + |\nabla u|^2)(5, \vec{x}) \ d\vec{x} \ = \ 10 \ .$$

Is it possible that at the initial time one could have:

$$E_{B_{1+5c}}(u)(0) \;=\; \frac{1}{2}\iint_{x^2+y^2+z^2\leqslant(1+5c)^2} (g^2 + |\nabla f|^2)\, d\vec{x} \;=\; 9 \ .$$

Please give a precise reason for your answer.

Problem 5: Using the three dimensional Kirchhoff formula, consider the values for the solution to the wave equation:

$$\partial_t^2 u - \Delta u \;=\; 0 \ , \qquad \text{On } \mathbb{R}\times\mathbb{R}^3 \ ,$$

$$u(0,\vec{x}) \;=\; 0 \ ,$$

$$u_t(0,\vec{x}) \;=\; \begin{cases} \sin(\pi r) \ , & 0 \leqslant r \leqslant 1 \ ; \\ 0 \ , & \text{otherwise} \ , \end{cases}$$

Please answer the following:

- Show that for all times $t \geqslant 0$, one has that $u(t,\vec{x}) \geqslant 0$.
- Compute the solution at the space time points $(\vec{x},t)$: $(1,0,0,3)$, $(2,2,2,1)$, and $(0,0,0,2)$. In fact compute $u(t,0,0,0)$ for $1 \leqslant t$ (as is the usual notation, the order of the time and space variables are switched inside the function).
- Compute the energy of $u(t,x,y,z)$ at time $t = 10$.

Problem 6: Recall that in class we showed if u is a solution to the three dimensional wave equation:

$$\partial_t^2 u - \Delta u \;=\; 0 \ , \qquad \text{On } \mathbb{R}\times\mathbb{R}^3 \ ,$$

$$u(0,\vec{x}) \;=\; f(\vec{x}) \ ,$$

$$u_t(0,\vec{x}) \;=\; g(\vec{x}) \ ,$$

then one has the *uniform* bound:

$$|u(t,\vec{x})| \;\leqslant\; \frac{C}{t} \ ,$$

where the constant C depends only on the initial data f and g. Clearly the same bound is also true for the functions u_t, u_x, u_y, and u_z, as these also solve the wave equation with initial data g and Δf (in the case of u_t) and so on.

However, assume that one could prove a better estimate like:

$$|u_t(t,\vec{x})| \;\leqslant\; \frac{C}{t^2} \ ,$$

for $t \to \infty$. Show that this kind of bound is impossible because it would violate the conservation of energy. For simplicity, you may assume that the initial data f and g are such that:

$$f(\vec{x}) \ = \ g(\vec{x}) \ = \ 0 \ , \qquad\qquad |\vec{x}| \ > \ 1 \ .$$

Problem 7: Use the two (spatial) dimensional Kirchhoff formula to show that if u is a solution to the Cauchy problem:

$$\begin{aligned} \partial_t^2 u - \Delta u \ &= \ 0 \ , \qquad\qquad \text{On } \mathbb{R}\times\mathbb{R}^2 \ , \\ u(0,\vec{x}) \ &= \ 0 \ , \\ u_t(0,\vec{x}) \ &= \ g(\vec{x}) \ , \end{aligned}$$

where we assume that g is bounded and integrable, then as $t \to \infty$ one has the simple asymptotic behavior:

$$u(t,0,0) \ \sim \ \frac{1}{2\pi c^2 t}\cdot \iint_{\mathbb{R}^2} g(x,y)\ dxdy \ .$$

In particular, such a solution can be strictly non-zero as $t \to \infty$ when evaluated at the spatial origin $\vec{x}_0 = (0,0)$. Notice that this phenomena is very different from the situation in 3 spatial dimensions.

This section concerns the heat equation.

Problem 1:

Use the formula for the solution to the $3D$ heat equation to show <u>infinite</u> speed of propagation. Specifically, show that if $u(t,x,y,z)$ solves the problem:

$$\begin{aligned} \partial_t u \ &= \ \Delta u \ , \\ u(0,x,y,z) \ &= \ \begin{cases} 1-r^2 \ , & r \leqslant 1; \\ 0 \ , & r > 1, \end{cases} \end{aligned}$$

then we have $u(t,x,y,z) > 0$ for *any* fixed point (t,x,y,z) with $t > 0$.

Problem 2: (Propagation Estimates) This problem is about understanding the spreading properties of solutions to the heat equation. Try to show that the solution from **Problem 1** obeys the following inequality:

$$\Big(\frac{1}{1+t}\Big)^{\frac{3}{2}} e^{-\frac{(x^2+y^2+z^2)}{4t}} \ < \ u(t,x,y,z) \ < \ \frac{8\pi}{15(4\pi t)^{\frac{3}{2}}} \ ,$$

for any $t > 0$. Show that these bounds are essentially sharp in the sense that $u(t,x,y,z)$ is exponentially small outside the ball of radius 1 for small times, and that for bounded values of (x,y,z) as $t \to \infty$ one has $u(t,x,y,z) \sim t^{-\frac{3}{2}}$. (Hint: Us the fact that the solution is rotationally symmetric to reduce the first inequality to the case of $(x,0,0)$. Its messy, but you should be able to get something via explicit integration).

Problem 3: Let $u(t,x,y,z)$ solve the three dimensional heat equation:

$$\begin{aligned} \partial_t u &= k\Delta u \ , \\ u(0,x,y,z) &= f(x,y,z) \ . \end{aligned}$$

Define the *total heat* to be the value:

$$H(t) \ = \ \int_{\mathbb{R}^3} u(t,x,y,z)\, dxdydz \ .$$

Show the total heat is conserved in the sense that:

$$H(t) \ = \ \int_{\mathbb{R}^3} f(x,y,z)\, dxdydz \ .$$