THE HILBERT SCHEME OF POINTS IN THE PLANE IS CONNECTED

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This is a difficult topic to summarize briefly. I decided to assume a fair amount of background and leave numerous definitions as “black boxes” while still making literally true claims. The general discussion largely follows the intro of [Nak99] as well as parts of Lothar Göttscbe’s segment of [FGI+05]. The final examples are discussed in more detail in [MS05]. Grothendieck’s original construction is in [Gro62].

1. Hilbert Schemes

Definition 1.1. Let $X$ be a projective scheme over an algebraically closed field $k$. The Hilbert scheme of $X$ is the scheme $\text{Hilb}_X$ characterized by the following property. The set of $k$-scheme morphisms $U \to \text{Hilb}_X$ is in bijection with closed $k$-subschemes $Z \subset X \times U$ such that the induced projection map $Z \to U$ is flat. Moreover, this bijection is contravariantly functorial.

In fact, $\text{Hilb}_X$ is projective. The case $U = \text{Spec } k$ is instructive, since then $Z$ is literally the set of closed $k$-subschemes of $X$. In this context the $k$-rational points and the closed points of $\text{Hilb}_X$ coincide. This is a rigorous version of the slogan that the Hilbert scheme of $X$ parameterizes the closed subschemes of $X$. For general $U$, we intuitively imagine $Z$ to be the “graph” of a continuously deforming family of subschemes of $X$.

We can refine the construction in two distinct ways. First, for each $Z$ above, we can associate a Hilbert polynomial $P(t) \in \mathbb{Q}[t]$ to each fiber $\pi^{-1}(u)$. We may then restrict to using only those $Z$ above whose fibers have a fixed Hilbert polynomial $P(t)$, resulting in the closed subscheme $\text{Hilb}^P_X$. On the other hand, we may pick an open subscheme $Y$ of $X$ and restrict to $Z \subset Y \times U$ above, resulting in an open subscheme $\text{Hilb}^P_Y$ of $\text{Hilb}^P_X$.

Definition 1.2. Let $X$ be a quasiprojective $k$-scheme. Let $P(t) = n$ be constant. The Hilbert scheme of $n$ points in $X$ is

$$X^{[n]} := \text{Hilb}^P_X.$$ 

The name arises from the fact that, given $n$ closed points $x_1, \ldots, x_n \in X$, the closed subscheme $Z = \{x_1, \ldots, x_n\}$ is a closed point of $X^{[n]}$. Indeed, these points form an open subset of $X^{[n]}$. As we will see, the “interesting” part of the Hilbert scheme is the complement of this set.

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2. A First Order Approximation

**Definition 2.1.** Let $X$ be a quasiprojective $k$-variety. The $n$th symmetric product of $X$ is the quasiprojective $k$-variety informally described as

$$X^{(n)} := X \times \cdots \times X / S_n$$

where $S_n$ acts by permuting factors. If $X$ is affine, this is the $k$-scheme $\text{Spec}((k[X]^\otimes n)^{S_n})$ where $k[X]$ is the coordinate ring of $X$.

It can be shown that $X^{(n)}$ is the geometric quotient of $X$ by $S_n$, i.e. that it satisfies an appropriate universal property. By definition, the orbits of $X^{(n)}$ are orbits of $n$-tuples of elements of $X$. This gives rise to a stratification

$$X^{(n)} = \bigsqcup_{\nu \vdash n} X^{(\nu)}$$

where $X^{(\nu)}$ consists of those multisubsets of $n$ elements of $X$ whose multiplicities are described by $\nu$. When $\nu = (1^n)$ we recover $n$-element subsets of $X$, which is an open subset of $X^{(n)}$. Informally, $X^{(n)}$ is a “first order” approximation to a moduli space of $n$ points in $X$. Geometric quotients like the above “rarely” exist as schemes while $X^{[n]}$ is a special case of a vastly more general construction.

**Theorem 2.2.** Let $X$ be a smooth quasiprojective variety over $k$. There is a surjective morphism of $k$-schemes

$$\pi : X^{[n]}_{\text{red}} \to X^{(n)}$$

called the Hilbert-Chow morphism, given on the level of points by

$$\pi(Z) := \sum_{x \in X} \text{length}(Z_x)[x].$$

In fact, $\pi$ is an isomorphism from $X^{((1^n))}$ to its inverse image. In this sense, the Hilbert scheme $X^{[n]}$ and the first order approximation $X^{(n)}$ agree “except at the edges.”

3. The First Two Interesting Cases

When $X$ is a smooth quasiprojective curve, we have $X^{[n]} = X^{(n)}$. For instance, the Hilbert scheme of points in the line $X = \mathbb{A} := \text{Spec} k$ is informally

$$\mathbb{A}^{[n]} = \{ I \subset k[t] \mid \text{I is an ideal, dim}_k k[t]/I = n \}$$

$$= \{ f(z) \in k[z] \mid z^n + a_1 z^{n-1} + \cdots + a_n, a_i \in k \}$$

$$= \mathbb{A}^{(n)}.$$

This is perhaps the first non-trivial example of a Hilbert scheme: we’ve used the second-simplest possible variety (a line instead of a point) and we’ve used the simplest family of Hilbert polynomials.

The second non-trivial example is then $(\mathbb{A}^2)^{[n]}$, the Hilbert scheme of $n$ points in the plane. Indeed, a theorem of Fogarty says that if $X$ is a smooth quasiprojective surface, then $X^{[n]}$ is smooth and irreducible, and $\pi$ is a resolution of singularities. In fact, $X^{(n)}$ in this latter case
is not smooth for \( n \geq 2 \), so in this case the two constructions genuinely differ and \( X^{[n]} \) is “better-behaved.”

The geometry of \( (\mathbb{A}^2)^{[n]} \) is extraordinarily rich and connects immediately to the combinatorics of Young diagrams. To see this, note that, at the level of closed points,
\[
(\mathbb{A}^2)^{[n]} = \{ I \subset k[x, y] \mid I \text{ is an ideal, } \dim_k k[x, y]/I = n \}.
\]
Fixing, say, lexicographic ordering with \( x > y \), we have
\[
\dim k[x, y]/I = \dim k[x, y]/\text{LT}(I).
\]
It is easy to see that the exponents \((a, b)\) of the monomials \( x^a y^b \) not in \( \text{LT}(I) \) form a lower order ideal in \( \mathbb{Z}^2_{\geq 0} \) under the component-wise partial order, consisting of \( n \) elements. This is precisely a Young diagram with \( n \) boxes. Moreover every such Young diagram arises in this way for some unique monomial ideal \( I_\lambda \) (generated by the “outer corners”).

Thus to each point of \( (\mathbb{A}^2)^{[n]} \) we may associate an ideal \( I_\lambda \), and in fact we may “continuously vary” the ideals in an appropriate sense to travel from \( I \) to \( I_\lambda \) inside \( (\mathbb{A}^2)^{[n]} \). Since \( V(I_\lambda) = \{(0, 0)\} \) for all \( \lambda \), the subscheme associated to \( I_\lambda \) is highly non-reduced, i.e. it is very much not the coordinate ring of a classical affine variety. Intuitively, it tracks extra information about the collisions of points as they all go to zero. Since we may also continuously deform \( I \in (\mathbb{A}^2)^{[n]} \) which vanish at \( n \) distinct points into each other, we have informally arrived at the fact that \( (\mathbb{A}^2)^{[n]} \) is connected.

Interestingly, some of the nice properties of the Hilbert scheme in dimension 1 and 2 begin to fail already at \( n = 3 \). For instance, the Hilbert-Chow morphism need not be a resolution of singularities, and the Hilbert scheme may have “unexpectedly” large dimension. It would nonetheless be interesting to see more combinatorial connections between plane partitions and \( (\mathbb{A}^3)^{[n]} \), since plane partitions index the relevant monomial ideals. Everything I found in this direction concerned Haiman’s use of \( (\mathbb{A}^2)^{[n]} \) for the \( n! \) conjecture.

References


