1. Primary goal

Our basic definition is the following:

**Definition 1** (Reiner-Stanton-White). Given \((W, C_n, f(q))\) where \(W\) is a finite \(C_n\)-set, and \(f(q) \in \mathbb{N}[q]\), this triple exhibits the cyclic sieving phenomenon if

\[
    f(\omega_n^k) = \# \{ w \in W : \sigma_n^k \cdot w = w \}
\]

where \(C_n = \langle \sigma_n \rangle\) and \(\omega_n \in \mathbb{C}\) is a primitive \(n\)th root of unity.

Our primary goal is to prove the following theorem:

**Theorem 2** ([RSW04] Thm. 1.6). Let \((W, S)\) be a finite Coxeter system and \(J \subset S\). Let \(C\) be a cyclic subgroup generated by a regular element of \(W\). Let \(W^J\) be the set of minimal length coset representatives for \(W/W_J\). Define

\[
    W^J(q) := \sum_{w \in W^J} q^{\ell(w)}.
\]

Then the triple

\((W, C, W^J(q))\)

exhibits the cyclic sieving phenomenon (CSP).

(Why? In tomorrow’s talk, we’ll generalize the type A case when \(C = \langle (1 2 \cdots n) \rangle\).)

A down-to-earth special case:

**Corollary 3.** Let \(\mathbb{Z}/n\) act on \(\binom{\mathbb{Z}/n}{k}\) naturally. Then

\[
    \left( \binom{\mathbb{Z}/n}{k}, \mathbb{Z}/n, \binom{n}{k}_q \right)
\]

exhibits the CSP.

**Remark 4.** The following are regular for any \((W, S)\):
• the longest element \( w_0 \);
• any Coxeter element, i.e. the product \( \prod_{s \in S} s \) taken in any order, all of which are conjugate.

In type \( A_{n-1} \), \( w_0 = n(n - 1) \cdots 21 \in S_n \), and the Coxeter elements are the \( n \)-cycles. Indeed, the regular elements in \( S_n \) are precisely the elements with cycle types of the form \( (a^b) \) or \( (a^b,1) \).

2. Complex reflection groups

We first deduce the preceding theorem from a more general result. We give some background before stating it.

**Definition 5.** \( w \in GL(\mathbb{C}^n) \) is a (pseudo-)reflection if \( |w| < \infty \) and the stabilizer of \( w \) has codimension 1; this stabilizer is called a reflecting hyperplane.

A (complex) reflection group is a finite subgroup \( W \) of \( GL(\mathbb{C}^n) \) generated by reflections. The coinvariant algebra of \( W \) is
\[
A := \mathbb{C}[x_1, \ldots, x_n]/I_W^I
\]
where
\[
I_W^I := (p: W \cdot p = \{p\}, p(0, \ldots, 0) = 0)
\]
and \( w \cdot p(x_1, \ldots, x_n) = p(w \cdot x_1, \ldots, w \cdot x_n) \) using \( \mathbb{C}^n = \langle e_1, \ldots, e_n \rangle \cong \mathbb{C} x_1 \oplus \cdots \oplus \mathbb{C} x_n \). This is a graded \( \mathbb{C} \)-algebra and a graded \( W \)-module.

Note that each \( w \in W \) is diagonalizable. An element \( w \in W \) is regular if some eigenvector of \( w \) does not lie in any of the reflecting hyperplanes of the reflections of \( W \).

**Example 6.** Let \( W = S_n \subset GL(\mathbb{C}^n) \) be given by permutation matrices. \( W \) is generated by \( s_i = (i \ i + 1) \) for \( 1 \leq i < n \) which has reflecting hyperplane \( x_i = x_{i+1} \). The reflections of \( W \) are the transpositions \( (i \ j) \) for \( 1 \leq i < j \leq n \) with reflecting hyperplane \( x_i = x_j \). The eigenspaces of \( h = (1 \ 2 \ \cdots \ n) \) are spanned by \( (1, \omega^k, \ldots, (\omega^k)^{n-1}) \), so \( h \) is regular.

\( I_W^I \) is generated by the homogeneous symmetric polynomials of positive degree, so \( I_W^I = \langle e_1, e_2, \ldots, e_n \rangle \).

It is not immediately obvious, but a basis for the coinvariant algebra \( A \) is given by the “staircase monomials” \( x_1^{a_1} \cdots x_n^{a_n} : a_i \leq n - i \), and in particular \( \dim \mathbb{C} A = n! \).

**Definition 7.** Let \( R = \oplus_{i \geq 0} R_i \) be a graded \( \mathbb{C} \)-algebra. Its Hilbert series is
\[
\text{Hilb}(R; q) := \sum_{i \geq 0} (\dim \mathbb{C} R_i) q^i.
\]

**Theorem 8** ([RSW04 Thm. 8.2]). Let \( W \) be a complex reflection group, \( \sigma \in W \) regular, \( W' \leq W \) any subgroup, \( A^{W'} \) the \( W' \)-invariant (graded) subalgebra of the coinvariant algebra of \( W \).

Then
\[
(W/W', \langle c \rangle, \text{Hilb}(A^{W'}, q))
\]
exhibits the CSP.

We can deduce the first theorem from the second using the following two observations. First, finite Coxeter groups are precisely those complex reflection groups which arise from extending scalars for real reflection groups. Second:

**Fact 9** ([Hil82 §IV.4]). Let \( (W, S) \) be a finite Coxeter system, \( J \subset S \). Then
\[
\text{Hilb}(A^{W_j}, q) = \sum_{w \in W^J} q^{\ell(w)} = W^J(q).
\]
3. RSW Proof

For the second theorem, the key is a generalization of the following well-known result:

**Theorem 10** ([Che55]). Let \( W \) be a complex reflection group. Then \( A \cong CW \) as \( W \)-modules.

**Theorem 11** ([Spr74, Prop. 4.5], cf. [KW01]). Let \( W \) be a complex reflection group, \( c \in W \) regular of order \( n \), and \( A \) the coinvariant algebra of \( W \). Let \( A \) be a \( W \times C \)-module via
\[
c \cdot x_i := \omega_n^i x_i
\]
for a fixed primitive \( n \)th root of unity \( \omega_n \). Let \( CW \) be a \( W \times C \)-module via
\[
(w, c) \cdot u = wuc.
\]
Then \( A \cong CW \) as \( W \times C \)-modules.

Given this, we’ll now sketch the proof of the second theorem:

**Proof.** Consider \((W, C_n, f(q))\) with \( C_n = \langle \sigma_n \rangle \). Let \( \chi^i : C_n \to \mathbb{C}^\times \) via \( \chi^i(\omega_n) := \omega_n^i \).

It’s straightforward to check that \((W, C_n, f(q))\) exhibits the CSP if and only if
\[
f(q) \equiv \sum_{i=0}^{n-1} a_i q^i \quad (\text{mod } q^n - 1)
\]
where \( a_i \) is the multiplicity of \( \chi^i \) in the \( C_n \)-module \( \mathbb{C}[W] \). (Note: \( \mathbb{C}[W] \) is the \( \mathbb{C} \)-vector space with basis \( W \) and the induced \( C \)-module action; \( CW \) is the group algebra of \( W \).

A restatement of this observation is the following. Let \( X = \oplus_{i=0}^{n-1} X_i \) be the \( \mathbb{C} \)-vector space where \( \dim X_i = a_i \) and \( \sigma_n \) acts on \( X_i \) as multiplication by \( \omega_n^i \). The triple \((W, C_n, f(q))\) exhibits the CSP if and only if \( W \cong X \) as \( C_n \)-modules.

Now consider \((W/W', \langle c \rangle, \text{Hilb}(A^{W'}, q))\). By Springer’s theorem, \( A \cong CW \) as \( W \times C \)-modules, so \( A^{W'} \cong (CW)^{W'} \) as \( C \)-modules. Notice that \( \omega_n \) acts on the \( i \)th component of \( A^{W'} \) by multiplication by \( \omega_n^i \). It follows that \( A^{W'} \cong X \) as \( C \)-modules. So, we must only show \( (CW)^{W'} \cong \mathbb{C}[W/W'] \) as \( C \)-modules. Indeed, it’s straightforward to check that
\[
\Phi : \mathbb{C}[W/W'] \to (CW)^{W'}
\]
given by
\[
\Phi(wW') := \sum_{u \in wW'} u
\]
gives an isomorphism of \( C \)-modules. (Technically, \( (CW)^{W'} \) are the right \( W' \)-invariants here, but \( CW \) is abelian, so it’s not an issue.) \( \square \)

4. Springer’s Result

It takes some effort to translate Springer’s actual result into the theorem above. This step doesn’t seem to have been written down (e.g. it’s not in [RSW04]), but here’s my write up. We first state Springer’s result in his language and then describe the translation.
Definition 12. Let $G$ be a complex reflection group. Suppose $\rho$ is an irreducible complex representation of $G$ with character $\chi$. Let $A = \bigoplus_{i \geq 0} A_i$ be the coinvariant algebra of $G$ graded by degree. The $\chi$-exponents of $G$ form the multiset $\{p_j(\chi)\}_j$ of degrees of the copies of $\chi$ in $A$.

For instance, if $\langle \chi, A_1 \rangle = 2$, $\langle \chi, A_2 \rangle = 3$, and the rest are 0, then $\{p_j(\chi)\} = \{1, 1, 2, 2, 2\}$.

Proposition 13 ([Spr74] Prop. 4.5). Let $G$ be a complex reflection group. Suppose $g \in G$ has an eigenvector with eigenvalue $\zeta$ contained in no reflecting hyperplane of $G$. Let $\rho$ be an irreducible complex representation of $G$, with character $\chi$. Then the eigenvalues of $g$ in the representation $\rho$ are the $\zeta^{-p_j(\chi)}$, where the $p_j(\chi)$ are the $\chi$-exponents of $G$.

We now show the two results’ relatively straightforward equivalence.

Remark 14. Continue the notation of Springer’s result. A consequence of [Spr74] Thm. 4.2 is that $\zeta$ above has the same multiplicative order as $g$, call it $d$.

Let $C := \langle g \rangle$ and set

$$\chi^i : C \to \mathbb{C}^\times$$

$$g \mapsto \zeta^{-i}.$$ for $0 \leq i < d$. Note that the irreducible representations of $G \times C$ are precisely of the form $\rho \boxtimes \chi^i$ for some unique irreducible $\rho$ of $G$ and $i$, where

$$(\rho \boxtimes \chi^i)(w, c) := \chi^i(c)\rho(w).$$

The eigendecomposition of $\rho(g)$ gives the irreducible decomposition of $\rho\uparrow_G^G$. A restatement of Springer’s result is then:

$$\langle \chi^i, \rho\uparrow_G^G \rangle = \text{multiplicity of } \zeta^{-i} \text{ in } \{\zeta^{-p_j(\chi)}\} = \sum_{j : j \equiv d_i} \langle \rho, A_j \rangle.$$ 

Giving $A_j$ the above $G \times C$-action,

$$(w, g) \cdot p(x_1, \ldots, x_n) := p(w \cdot \zeta x_1, \ldots, w \cdot \zeta x_n),$$

we again find that $g$ acts on $A_j$ as multiplication by $\zeta^j$. Hence any $G$-submodule of $A_j$ is indeed a $G \times C$-submodule. It follows that

$$\langle \rho \boxtimes \chi^i, A_j \rangle = \begin{cases} \langle \rho, A_j \rangle & \text{if } i \equiv d_j \\ 0 & \text{otherwise.} \end{cases}$$

Hence another restatement of Springer’s result is

$$\langle \chi^i, \rho\uparrow_G^G \rangle = \langle \rho \boxtimes \chi^i, A \rangle.$$ 

On the other hand, write $CG$ to denote $CG$ with the above $G \times C$-action,

$$(w, g) \cdot u := wug.$$ 

Hence $CG \cong A$ if and only if

$$\langle \rho \boxtimes \chi^i, A \rangle = \langle \rho \boxtimes \chi^i, CG \rangle.$$ 

We’ll directly show

$$\langle \chi^i\uparrow_C \rho, \rangle = \langle \rho \boxtimes \chi^i, CG \rangle$$

which, using Frobenius reciprocity, will establish the equivalence of Springer’s result and the RSW restatement.

First, note that as left $C$-modules

$$CG \cong \bigoplus_{j=0}^{d-1} \chi^j$$

so that as left $G$-modules

$$CG \cong CC\uparrow_G^G = CG \otimes_{CC} CC \cong \bigoplus_{j=0}^{d-1} (CG \otimes_{CC} \chi^j).$$
Write $\chi^j \uparrow^G_C$ to denote $\chi^j \uparrow^G_C$ with the $G \times C$-action

$$(w, g) \cdot u \otimes v := (wu) \otimes (g \cdot v) = \zeta^j (wu \otimes v)$$

The preceding $G$-module isomorphism then yields a $G \times C$-module isomorphism

$$C G \cong \oplus_{j=0}^{d-1} \chi^j \uparrow^G_C.$$

Hence it suffices to show

$$\langle \rho \boxtimes \chi^i, \chi^j \uparrow^G_C \rangle = \delta_{ij} \langle \rho, \chi^j \uparrow^G_C \rangle.$$

Since $g$ acts on $\chi^j \uparrow^G_C$ as multiplication by $\zeta^j$, this follows as before for $\langle \rho \boxtimes \chi^i, A_j \rangle$.

References