IRREDUCIBLE DECOMPOSITIONS

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(This lecture was given in the Applied Algebraic Geometry topics course at the University of Washington on May 1st, 2017. It essentially follows [CLO15, §4.6].)

1. Irreducible Decompositions

Example 1.1. We have \( V(xy) = V(x) \cup V(y) \). Can we break \( V(x) \) down further, usefully? (Draw.) We give a formal generalization of this example today.

Remark 1.2. Recall the following version of Hilbert’s Basis Theorem. Every ascending chain of ideals

\[ I_1 \subset I_2 \subset \cdots \subset k[x_1, \ldots, x_n] \]

eventually stabilizes, i.e. \( \exists N \) such that \( I_N = I_{N+1} = \cdots \).

Corollary 1.3. Any descending chain of affine varieties

\[ V_1 \supset V_2 \supset \cdots \supset k^n \]

eventually stabilizes, i.e. \( \exists N \) such that \( V_N = V_{N+1} = \cdots \).

Proof. Apply \( I \), then \( V \), noting \( V(I(V_i)) = \overline{V_i} = V_i \). \( \square \)

Theorem 1.4. Let \( V \subset k^n \) be an affine variety. Then \( V \) can be written as a finite union

\[ V = V_1 \cup \cdots \cup V_m \]

where each \( V_i \) is irreducible.

Proof. If \( V \) is irreducible, then we’re done, so suppose \( V \) is reducible. Write \( V = V_1 \cup V'_1 \) where \( V_1, V'_1 \) are varieties such that \( V_1 \neq V \neq V'_1 \). If \( V_1, V'_1 \) are each irreducible, then we’re done, so suppose \( V_1 \) is reducible. Hence write \( V_1 = V_2 \cup V'_2 \) with \( V_2 \neq V'_2 \). Repeating this, we either terminate eventually or we get an infinite chain

\[ V \supseteq V_1 \supseteq V_2 \supseteq \cdots , \]
a contradiction. \( \square \)

Example 1.5. We have \( V(xy) = V(x) \cup V(y) \cup V(x, y) \), but \( V(x, y) \) is “unnecessary.”

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Theorem 1.6. Let \( V \subset k^n \) be an affine variety. Then \( V \) has a minimal decomposition
\[
V = V_1 \cup \cdots \cup V_m
\]
where each \( V_i \) is an irreducible variety with \( V_i \nsubseteq V_j \) for all \( I \neq j \). Up to reordering, this decomposition is unique.

Proof. Existence follows from the previous theorem after removing parts contained in others. For uniqueness, suppose we have another minimal decomposition
\[
V = V'_1 \cup \cdots \cup V'_l.
\]
Fix \( i \) and note
\[
V_i = V_i \cap V = V \cap (V'_1 \cup \cdots \cup V'_l) = (V \cap V'_1) \cup \cdots \cup (V \cap V'_l).
\]
Since \( V_i \) is irreducible, \( V_i = V_i \cap V'_j \) for some \( j \), so \( V_i \subset V'_j \). By the symmetry of this argument, we have some \( k \) such that \( V'_j \subset V_k \). Hence \( V_i \subset V_k \), so \( i = k \) and \( V_i = V'_j \). This procedure results in a bijection
\[
\{V_1, \ldots, V_m\} \leftrightarrow \{V'_1, \ldots, V'_l\}.
\]
In particular, \( l = m \) and the result follows. \( \square \)

Corollary 1.7. If \( k = \overline{k}, \) every \( I = \sqrt{I} \) has a minimal decomposition
\[
I = P_1 \cap \cdots \cap P_r
\]
for \( P_i \) prime with \( P_i \nsubseteq P_j \) for all \( i \neq j \).

Proof. Apply the \( I \) and \( V \) bijections. \( \square \)

Remark 1.8. In fact, the corollary holds even if \( k \neq \overline{k} \).

Theorem 1.9. If we have a minimal decomposition
\[
I = \sqrt{I} = P_1 \cap \cdots \cap P_r \subset k[x_1, \ldots, x_n]
\]
with \( P_i \) prime, \( P_i \nsubseteq P_j \) for all \( i \neq j \). Then
\[
\{P_1, \ldots, P_r\} = \{I : f \text{ proper, prime } | f \in k[x_1, \ldots, x_n]\}.
\]

Proof. First, some ingredients.

<table>
<thead>
<tr>
<th>Fact</th>
<th>Intuition</th>
</tr>
</thead>
<tbody>
<tr>
<td>( (\cap_{i=1}^n I_i) : J = \cap_{i=1}^n (I_i : J) )</td>
<td>( (\cup_i V_i) - V = \cup_i (V_i - V) )</td>
</tr>
<tr>
<td>( f \in P \Rightarrow P : f = {1} )</td>
<td>( W := V(f) \supseteq V(P) ) implies ( V(P) - V(f) = \emptyset )</td>
</tr>
<tr>
<td>( f \notin P \Rightarrow P : f = P )</td>
<td>( W := V(f) \nsubseteq V(P) ) implies ( V(P) - W = V(P) )</td>
</tr>
<tr>
<td>If ( P = \cap_{i=1}^n I_i ), then ( P = I_i ) for some ( i )</td>
<td>( V = V_1 \cup \cdots \cup V_n ) implies ( V = V_i ) for some ( i )</td>
</tr>
</tbody>
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Now, for \( \supseteq \), suppose \( I : f \) is proper and prime. First by (1),
\[
I : f = (\cap_i P_i) : f = \cap_i (P_i : f).
\]
By (4) since \( I : f \) is prime, \( I : f = P_i : f \) for some \( i \). By (2) and (3) since \( I : f \) is proper, \( P_i : f = P_i \). Hence \( I : f = P_i : f = P_i \).

For \( \subset \), fix \( i \) and pick \( f \in (\cap_{j \neq i} P_j) - P_i \), which is \( \neq \emptyset \) by minimality. Hence by (3), \( P_i : f = P_i \), and by (2), \( P_i : f = \{1\} \) for all \( j \neq i \). Thus \( I : f = P_i \). \( \square \)
Example 1.10. One may verify the following irreducible decomposition.

\[ I = (xz - y^2, x^3 - yz) \]
\[ = (x, y) \cap (xz - y^2, x^3 - yz, x^2y - z^2) \]
\[ = (I : x^2y - z^2) \cap (I : x). \]

Remark 1.11. There are algorithms for

- deciding if an ideal is prime, or if an affine variety is irreducible;
- finding the irreducible decomposition of a variety or radical ideal.

They’re not discussed in [CLO15]; see the references at the end of [CLO15, §4.6]

References