Major Index Asymptotics
AMS Special Session on Combinatorial Representation Theory,
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based partly on joint work with
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arXiv:1701.04963
Question (Sundaram)

Fix an $S_n$-conjugacy class $\mu$. Let $S_n$ act by conjugation on $\mu$ $\mathbb{C}$-linearly. For which $\mu$ does every $S_n$-irreducible appear in this representation?
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Global conjugacy classes

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- When $\mu = (n)$, this representation is $1^{\uparrow S_n}_{C_n}$ where $C_n := \langle (\sigma_n) \rangle$ with $\sigma_n := (1 \ 2 \ \cdots \ n)$
Combinatorial representation theory

Define $\chi^r : C_n \to \mathbb{C}^\times$ by $\chi^r(\sigma^k_n) := \omega_n^{kr}$
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- Define $\chi^r : C_n \to \mathbb{C}^\times$ by $\chi^r(\sigma_n^k) := \omega_n^{kr}$
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$$a_{\lambda,r} := \langle \chi^r_{\uparrow S_n}^{C_n}, \chi^\lambda \rangle$$
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**Theorem (Kraskiewicz–Weyman)**

*Let $\lambda \vdash n$. Then*

$$a_{\lambda, r} = \# \{ T \in SYT(\lambda) : \text{maj}(T) \equiv_n r \}.$$
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Theorem (Kraskiewicz–Weyman)

Let $\lambda \vdash n$. Then

$$a_{\lambda, r} = \# \{ T \in \text{SYT}(\lambda) : \text{maj}(T) \equiv_n r \}.$$

Related to work of Thrall, Klyachko, Stembridge, Lusztig, Stanley, ...
Major index on standard tableaux

- $\text{SYT}(\lambda/\nu) := \{\text{standard Young tableaux of shape } \lambda/\nu\}$
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- \( \text{SYT}(\lambda/\nu) := \{ \text{standard Young tableaux of shape } \lambda/\nu \} \)
- For \( T \in \text{SYT}(\lambda/\nu) \), set
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  \text{Des}(T) := \{ i : i + 1 \text{ appears in a lower row than } i \text{ in } T \}
  \]

Example: \( \lambda/\nu = (4,3,2)/(1) \), \( T = \begin{array}{ccc}
1 & 6 & 7 \\
2 & 4 & 8 \\
3 & 5 & \end{array} \) has \( \text{Des}(T) = \{ 1, 2, 4, 7 \} \) and \( \text{maj}(T) = 1 + 2 + 4 + 7 = 14 \)
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Restatement of earlier conjecture:

**Conjecture (Sundaram)**

Let $\lambda \vdash n > 1$. Then $a_{\lambda,0} = 0$ if and only if

- $\lambda = (n - 1, 1)$, or
- $\lambda = (2, 1^{n-2})$ if $n$ is odd, or
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Related earlier work:

**Theorem (Klyachko)**

Let $\lambda \vdash n > 1$. Then $a_{\lambda,1} = 0$ if and only if

- $\lambda = (2, 2)$, or $\lambda = (2, 2, 2)$, or
- $\lambda = (n)$, or
- $\lambda = (1^n)$ when $n > 2$
Estimating $a_{\lambda,r}$

Theorem (S.)

*For all* $\lambda \vdash n \geq 1$ *and all* $r$,

\[
\left| \frac{a_{\lambda,r}}{f^\lambda} - \frac{1}{n} \right| \leq \frac{2n^{3/2}}{\sqrt{f^\lambda}}
\]

Proof uses Foulkes' formula, Ramanujan sums, $\chi_{\lambda}(\ell n/\ell)$, the Fomin–Lulov bound, Stirling's approximation.
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Let $\lambda \vdash n \geq 81$ with $\lambda_1, \lambda'_1 < n - 7$. Then $f^\lambda \geq n^5$ and

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- Proof uses “opposite hook products” arising independently in recent work of Morales–Pak–Panova
Results for $a_{\lambda,r}$

Corollary (S.)

Let $\lambda \vdash n > 1$. Then $a_{\lambda,r} = 0$ if and only if

- $\lambda = (2,2)$, $r = 1,3$; or $\lambda = (2,2,2)$, $r = 1,5$; or $\lambda = (3,3)$, $r = 2,4$; or
- $\lambda = (n-1,1)$ and $r = 0$; or
- $\lambda = (2,1^{n-2})$, $r = \begin{cases} 0 & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even} \end{cases}$; or
- $\lambda = (n)$, $r \in \{1, \ldots, n-1\}$; or
- $\lambda = (1^n)$, $r \in \begin{cases} \{1, \ldots, n-1\} & \text{if } n \text{ is odd} \\ \{0, \ldots, n-1\} - \{\frac{n}{2}\} & \text{if } n \text{ is even}. \end{cases}$
Results for $a_{\lambda,r}$

Corollary

*Sundaram’s conjecture is true!*
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The statistic “maj mod n” is intuitively uniformly distributed as $n \to \infty$. 

Results for $a_{\lambda, r}$
Corollary

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The statistic “maj mod n” is intuitively uniformly distributed as $n \to \infty$.

Question

Can we remove “mod n”?
Asymptotic normality: definition

- Let $X$ be a random variable with mean $\mu$, variance $\sigma^2$. Let $X^* := (X - \mu)/\sigma$. 
Asymptotic normality: definition

- Let $X$ be a random variable with mean $\mu$, variance $\sigma^2$. Let $X^* := (X - \mu)/\sigma$.
- Let $X_1, X_2, \ldots$ be a sequence of random variables. Let $X^*_N$ have cumulative distribution function $F_N(t) := \mathbb{P}[X^*_N \leq t]$. 
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Definition

$X_1, X_2, \ldots$ is asymptotically normal if for all $t \in \mathbb{R}$,

$$\lim_{N \to \infty} F_N(t) = F(t)$$

where $F(t)$ is the CDF of the standard normal distribution.
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where $F(t)$ is the CDF of the standard normal distribution.

- If $X$ has a density function $f(t)$, the characteristic function $\mathbb{E}[e^{itX}]$ of $X$ is the Fourier transform of $f(t)$.
Asymptotic normality: examples

The “original” asymptotic normality result:

**Theorem (de Moivre, Laplace)**

*Let* $X_N$ *be the “cardinality” statistic on subsets of* $[N]$. *Then* $X_1, X_2, \ldots$ *is asymptotically normal.*
Asymptotic normality: examples

The “original” asymptotic normality result:

**Theorem (de Moivre, Laplace)**

Let $X_N$ be the “cardinality” statistic on subsets of $[N]$. Then $X_1, X_2, \ldots$ is asymptotically normal.

More generally:

**Theorem (Central limit theorem)**

Let $X_N$ be the average of $N$ i.i.d. random variables with finite variance. Then $X_1, X_2, \ldots$ is asymptotically normal.
Asymptotic normality: criteria

We can use characteristic functions:
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**Theorem (Lévy’s continuity theorem)**

A sequence $X_1, X_2, \ldots$ of random variables is asymptotically normal if and only if for all $t \in \mathbb{R}$,

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\lim_{N \to \infty} \mathbb{E}[e^{itX_N^*}] = e^{-t^2}
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There’s a classic, straightforward proof of the CLT using characteristic functions.
Asymptotic normality: criteria

Or, we can look at moments separately:

Theorem (Frechet–Shohat theorem)

A sequence $X_1, X_2, ...$ of random variables (with density functions that decay at least exponentially in the tails) is asymptotically normal if and only if for all $d \in \mathbb{Z} \geq 1$ we have

$$\lim_{N \to \infty} E[(X^*_N)^d] = \begin{cases} 0 & \text{if } d \text{ is odd} \\ \frac{(d-1)!!}{2^{d/2}} & \text{if } d \text{ is even} \end{cases}$$
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Definition
Let \( \text{aft}(\lambda) := |\lambda| - \max\{\lambda_1, \tilde{\lambda}_1\} \).
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Theorem (Billey–Konvalinka–S.)
Suppose $\lambda^{(1)}, \lambda^{(2)}, \ldots$ is a sequence of partitions. Let $X_N$ be the random variable corresponding to the major index statistic on $\text{SYT}(\lambda^{(N)})$. Then, the sequence $X_1, X_2, \ldots$ is asymptotically normal if and only if $aft(\lambda^{(N)}) \to \infty$ as $N \to \infty$. 

Asymptotic normality and standard tableaux
Asymptotic normality and standard tableaux

Example

\( \lambda^{(1)} = (50, 2), \ \text{aft}(\lambda^{(1)}) = 2 \)
Example

$\lambda^{(2)} = (50, 3, 1), \ \text{aft}(\lambda^{(2)}) = 4$
Asymptotic normality and standard tableaux

Example

\( \lambda^{(3)} = (8, 8, 7, 6, 5, 5, 5, 2, 2) \), \( \text{aft}(\lambda^{(3)}) = 39 \)
Asymptotic normality and standard tableaux

Corollary (Chen–Wang–Wang)

Using $\lambda^{(N)} = (N, N)$, the coefficients of the $q$-Catalan numbers

$$\frac{1}{[N+1]_q} \binom{2N}{N}_q$$

are asymptotically normal.
Corollary (Chen–Wang–Wang)

\( \lambda^{(N)} = (N, N) \), the coefficients of the \( q \)-Catalan numbers
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Proof of theorem uses cumulants, Stanley’s formula for \( \text{SYT}(\lambda)^{\text{maj}}(q) \), hook length estimates, method of moments
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- Proof of theorem uses cumulants, Stanley’s formula for SYT(\( \lambda \))\( \text{maj} \)(\( q \)), hook length estimates, method of moments
- SYT(\( \lambda \))\( \text{maj} \)(\( q \)) connects to principal specializations of \( s_\lambda \); type \( A \) coinvariant algebra and Lusztig–Stanley theorem; \( GL_n(\mathbb{F}_q) \)-representation theory by work of Green, Steinberg
Further work

- \( \text{diag}(\lambda) \) case settled; generalizes earlier work of Canfield–Janson–Zeilberger, Diaconis, Mann–Whitney, . . .

- Conjectured classification when \( \# \{ T \in \text{SYT}(\lambda) : \text{maj}(T) = k \} = 0 \)

- Similar conjectures for unimodality, log-concavity

- Progress towards a local limit theorem for \( \text{SYT}(\lambda) \)

- General skew shapes?
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Thanks!

FIN.


