Tableaux posets and the fake degrees of coinvariant algebras

AMS Western Sectional Meeting at SFSU

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based on joint work with
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slides: http://www.math.ucsd.edu/~jswanson/

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Outline

- Complex reflection groups and coinvariant algebras
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Fake degrees and internal zeros
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- Fake degrees and internal zeros
- Type A: rotation and block rules
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- Type A: rotation and block rules
- $G(m, 1, n)$ generalization
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- Complex reflection groups and coinvariant algebras
- Fake degrees and internal zeros
- Type A: rotation and block rules
- $G(m, 1, n)$ generalization
- $G(m, d, n)$ further generalization
Complex Reflection Groups

Definition

Let $V$ be a finite-dimensional complex vector space. $T \in \text{End}(V)$ is a \textit{pseudo-reflection} if $T$ has finite order and leaves a hyperplane fixed pointwise.
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Examples

- Dihedral groups (Type A)
- Symmetric groups (Type B)
- Hyperoctahedral groups (signed permutations; symmetries of the hypercube $\{\pm e_i\}$)
- Cyclic groups
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Given $m, n \in \mathbb{Z}_{\geq 1}$, let $G(m, 1, n)$ be the group of $n \times n$ pseudo-permutation matrices whose non-zero entries are from $C_m := \{\zeta \in \mathbb{C} : \zeta^m = 1\}$. 

Theorem (Shephard–Todd '53)

Up to isomorphism, the complex reflection groups are precisely the direct products of the groups $G(m, d, n)$ along with 34 exceptional groups.
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Definition

Given $d | m$, let $G(m, 1, n) \to C_m$ be given by multiplying the non-zero elements, let $\phi: G(m, 1, n) \to C_m \to C_d$, and set $G(m, d, n) := \ker \phi$. 

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Coinvariant Algebras

Definition

Given $G \leq \text{GL}(V)$, the *coinvariant algebra* of $G$ is

$$R_G := \frac{\text{Sym}(V)}{I_G}$$

where $I_G$ is the ideal generated by all homogeneous $G$-invariants of positive degree.

Remark

$R_G$ is a graded $G$-module.

(Type A) When $G = S_n$, $R_G = R_n = \mathbb{C}[x_1, \ldots, x_n]/\langle e_1, \ldots, e_n \rangle$.

(Type B) When $G = G(2, 1, n)$, $R_G = R_{2, 1, n} = \mathbb{C}[x_1, \ldots, x_n]/\langle e_i(x_2^i, \ldots, x_n^i) : 1 \leq i \leq n \rangle$. 

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Theorem

(Chevalley '55) $R_G$ as an ungraded module is isomorphic to the regular representation of the complex reflection group $G$. 

Fake Degrees

Definition

Let $S$ be an irreducible representation of $G$. Lusztig called $f_S(q) = \sum_{i \geq 0} \text{mult. of } S$ in $i$th degree piece of $R_G \cdot q^i$ the fake degree of $S$. 

By Chevalley’s result, $f_S(1) = \deg S$. 

Equivalently, what are the $f_S(q)$'s?
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Question What is the graded irreducible decomposition of $R_G$ for $G = G(m, d, n)$?
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Question

What is the graded irreducible decomposition of \( R_G \) for \( G = G(m, d, n) \)? Equivalently, what are the \( f^S(q) \)'s?
Partitions

Definition

A partition $\lambda$ of $n$ is a sequence of positive integers $\lambda_1 \geq \lambda_2 \geq \cdots$ such that $\sum_i \lambda_i = n$. 
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\[
\lambda = (5, 3, 1) \leftrightarrow \begin{array}{cccc}
  & & & \\
  & & & \\
  & & & \\
  & & & \\
  & & & \\
  & & & \\
  & & & \\
  & & & \\
  & & & \\
  & & & \\
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\square & \square & \square & \\
\square & \square & \\
\square & \\
\end{array}
\]

Theorem
(Young, early 1900’s) The complex inequivalent irreducible representations $S^\lambda$ of $S_n$ are canonically indexed by partitions of $n$.

Remark
By contrast, the irreps of $C_m$ are most naturally indexed by $\mathbb{Z}/m$ only up to $\phi(m)$ additive automorphisms.
Standard Tableaux

Definition

A standard Young tableau (SYT) of shape \( \lambda \vdash n \) is a filling of the cells of the Ferrers diagram of \( \lambda \) with 1, 2, \ldots, \( n \) which increases along rows and decreases down columns.

\[
T = \begin{array}{cccc}
1 & 3 & 6 & 7 & 9 \\
2 & 5 & 8 \\
4
\end{array} \in \text{SYT}(\lambda)
\]
Standard Tableaux

A *standard Young tableau* ($SYT$) of shape $\lambda \vdash n$ is a filling of the cells of the Ferrers diagram of $\lambda$ with $1, 2, \ldots, n$ which increases along rows and decreases down columns.

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\end{array} \in SYT(\lambda)$$

Descent set: $\{1, 3, 7\}$.

The *descent set* of $T \in SYT(\lambda)$ is the set

$$\{1 \leq i < n : i + 1 \text{ is in a lower row of } T \text{ than } i\}.$$
**Standard Tableaux**

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A *standard Young tableau (SYT)* of shape $\lambda \vdash n$ is a filling of the cells of the Ferrers diagram of $\lambda$ with $1, 2, \ldots, n$ which **increases along rows** and **decreases down columns**.

$T = \begin{array}{cccc}
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\end{array} \in \text{SYT}(\lambda)$

Descent set: $\{1, 3, 7\}$. Major index: $1 + 3 + 7 = 11$.

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The *descent set* of $T \in \text{SYT}(\lambda)$ is the set

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**Definition**

The *major index* of $T \in \text{SYT}(\lambda)$ is the sum of the descents.
Theorem

(Lusztig–Stanley ’70’s) The type A fake degrees are

\[ f^{S^\lambda}(q) = f^\lambda(q) = \sum_{T \in \text{SYT}(\lambda)} q^{\text{maj}(T)}. \]
Type A Fake Degrees

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Equivalently, the number of copies of \( S^\lambda \) in the \( r \)th graded piece of \( R_n \) is \( \#\{ T \in \text{SYT}(\lambda) : \text{maj}(T) = r \} \).
Type A Fake Degrees

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Example \( f^{(5,3,1)}(q) = q^5 (q^{18} + 2q^{17} + 4q^{16} + 5q^{15} + 8q^{14} + 10q^{13} + 13q^{12} + 14q^{11} + 16q^{10} + 16q^9 + 16q^8 + 14q^7 + 13q^6 + 10q^5 + 8q^4 + 5q^3 + 4q^2 + 2q + 1) \).
Type A Fake Degrees

Visualizing the coefficients of $q^{-5} f^{(5,3,1)}(q)$:

$(1, 2, 4, 5, 8, 10, 13, 14, 16, 16, 16, 14, 13, 10, 8, 5, 4, 2, 1)$
Enumerative Questions

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Are the fake degree coefficients log-concave?
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- Are they unimodal?
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- **When are they zero?**
Enumerative Questions

Question

- Are the fake degree coefficients log-concave?
- Are they unimodal?
- **When are they zero?** (Adin–Elizalde–Roichman recently and independently asked this question about the number of descents rather than maj.)
Type A Internal Zeros Classification

Lemma

(BKS 18+) The $b(\lambda) + 1$ coefficient of $f^\lambda(q)$ is zero if and only if $\lambda$ is a rectangle (not a row or column).
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\textbf{Are there other internal zeros?}
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(BKS 18+) The $b(\lambda) + 1$ coefficient of $f^\lambda(q)$ is zero if and only if $\lambda$ is a rectangle (not a row or column).

Question
Are there other internal zeros? No:

Theorem
(BKS 18+) The fake degree $f^\lambda(q)$ has internal zeros if and only if $\lambda$ is a rectangle (not a row or column).
Type A Internal Zeros Classification

Lemma (BKS 18+) *The* \(b(\lambda) + 1\) *coefficient of* \(f^\lambda(q)\) *is zero if and only if* \(\lambda\) *is a rectangle (not a row or column).*

Question Are there other internal zeros? No:

Theorem (BKS 18+) *The fake degree* \(f^\lambda(q)\) *has internal zeros if and only if* \(\lambda\) *is a rectangle (not a row or column).*

Corollary (Best Primality Test!) \(n > 1\) *is prime if and only if* \(f^\lambda(q)\) *has no internal zeros for any* \(\lambda \vdash n\).
Proof Strategy

- Start at the unique $T \in SYT(\lambda)$ with minimal maj.
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- Start at the unique $T \in \text{SYT}(\lambda)$ with minimal maj.
- Find a map $\phi: \text{SYT}(\lambda) - \mathcal{E}(\lambda) \rightarrow \text{SYT}(\lambda)$ which only slightly alters descent sets such that $\text{maj}(\phi(T)) = \text{maj}(T) + 1$. 

[Diagram or illustration related to proof strategy]
Proof Strategy

- Start at the unique $T \in \text{SYT}(\lambda)$ with minimal maj.
- Find a map $\phi : \text{SYT}(\lambda) - \mathcal{E}(\lambda) \to \text{SYT}(\lambda)$ which only slightly alters descent sets such that $\text{maj}(\phi(T)) = \text{maj}(T) + 1$.
- Iterate $\phi$ starting at $\text{minmaj}(\lambda)$, ending at $\text{maxmaj}(\lambda)$. 
Rotations

Definition

A positive rotation for \( T \in \text{SYT}(\lambda) \) is an interval \([i, k] \subset [n]\) such that if \( T' := (i, i + 1, \ldots, k - 1, k) \cdot T\), then \( T' \in \text{SYT}(\lambda) \) and there is some \( j \) for which

\[
\{j\} = \text{Des}(T') - \text{Des}(T)
\]

\[
\{j - 1\} = \text{Des}(T) - \text{Des}(T').
\]

A negative rotation is exactly the same except \((i, i + 1, \ldots, k - 1, k)\) is reversed.

Key Fact Applying rotations increases maj by 1!
Rotations

A *positive rotation* for $T \in SYT(\lambda)$ is an interval $[i, k] \subset [n]$ such that if $T' := (i, i + 1, \ldots, k - 1, k) \cdot T$, then $T' \in SYT(\lambda)$ and there is some $j$ for which

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A **negative rotation** is exactly the same except $(i, i + 1, \ldots, k - 1, k)$ is reversed.

**Key Fact** Applying rotations increases maj by 1!
Rotations

$T' := (i, i + 1, \ldots, k - 1, k) \cdot T$ or $T' := (k, k - 1, \ldots, i + 1, i) \cdot T$, $T' \in \text{SYT}(\lambda)$, $\exists j$ s.t. the descent at $j - 1$ in $T$ turned into a descent at $j$ in $T'$.

Example

\[
\begin{array}{ccc}
1 & 4 & \\
2 & 5 & \\
3 & 9 & \\
6 & & \\
7 & & \\
8 & & \\
\end{array}
\quad\rightarrow\quad
\begin{array}{ccc}
1 & 3 & \\
2 & 4 & \\
5 & 9 & \\
6 & & \\
7 & & \\
8 & & \\
\end{array}
\]

$\text{Des}(T) = \{1, 2, 4, 5, 6, 7\}$

$\rightarrow \text{Des}(T') = \{1, 3, 4, 5, 6, 7\}$
Rotations

Rotations have a characterization using combinatorial “patterns”
Rotations

- Rotations have a characterization using combinatorial “patterns”
- Rotations are plentiful: for SYT(5, 4, 4, 2), only 24 out of 81081 tableaux cannot be rotated

Lemma
Every (non-exceptional) tableau which avoids the pattern 1 2 /uni22EF i i + 1 z + 1 i + 2 ⋮ z admits a rotation. Rotations preserve the number of descents, but minmaj(λ) and maxmaj(λ) typically have different numbers of descents.
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\begin{array}{cccc}
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 & i+1 & z+1 & \\
 & i+2 & \\
 & \vdots & \\
 & z &
\end{array}
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\vdots & & & \vdots \\
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\end{array}
\]

admits a rotation.

Rotations preserve the number of descents, but \(\text{minmaj}(\lambda)\) and \(\text{maxmaj}(\lambda)\) typically have different numbers of descents.
Block Rules

We have 5 additional “block rules” which add a descent while incrementing maj by 1.

Example

B2:

```
1 2 3 4 5
6 7 8 9 10
11 12 13
```
Strong Poset

Each rotation rule and block rule has an “inverse-transpose” version obtained from the combinatorial descriptions by transposing the diagrams and reversing the arrows.
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The strong SYT poset $P(\lambda)$ on $\text{SYT}(\lambda)$ is obtained by defining the cover relations to be rotations, block rules, and their inverse-transposes.
Strong Poset

Each rotation rule and block rule has an “inverse-transpose” version obtained from the combinatorial descriptions by transposing the diagrams and reversing the arrows.

**Definition**

The *strong SYT poset* $P(\lambda)$ on $\text{SYT}(\lambda)$ is obtained by defining the cover relations to be rotations, block rules, and their inverse-transposes.

**Corollary**

If $\lambda$ is not a rectangle, $P(\lambda)$ is ranked (up to a shift) by $\text{maj}$ and has unique minimal and maximal elements. Indeed, $P(\lambda)$ is ranked by $(\text{des}, \text{maj} - \text{des})$ in the sense that rotation rules increase this by $(0, 1)$ and block rules increase this by $(1, 0)$. 
Strong Poset

Example

For SYT(3, 2, 1):
Corollaries

- Type A maj internal zeros classification
Corollaries

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- Answered Adin–Elizalde–Roichman des internal zeros question for straight shapes (there are none)
Corollaries

- Type A maj internal zeros classification
- Answered Adin–Elizalde–Roichman des internal zeros question for straight shapes (there are none)
- maj–des internal zeros classification for free
**G(m, 1, n) Fake Degrees**

(Specht, ’35) *The irreps of G(m, 1, n) are indexed (more-or-less canonically) by block diagonal skew partitions \( \lambda \) with \( m \) blocks and \( n \) total cells.*

**Example**

\( n = 10, m = 3: \)

\[ \lambda = ((3, 2), (1, 1), (3)) = \]

\[
\begin{array}{cccccccccc}
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
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\text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } & \text{ } \\
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\[ \lambda = ((3, 2), (1, 1), (3)) = \]

(The fake degrees are the same up to a \( q \)-shift regardless of the indexing scheme.)

(Stembridge ’89) *For \( \underline{\lambda} = (\lambda^{(1)}, \ldots, \lambda^{(m)}) \vdash n, \)

\[
f^{S\underline{\lambda}}(q) = f^{\underline{\lambda}}(q) = q^{b(\alpha(\underline{\lambda}))} \left( \frac{n}{\alpha(\lambda)} \right) \prod_{i=1}^{m} f^{\lambda^{(i)}}(q^m).\]
Theorem

(BKS 18+) Let $\lambda$ be a sequence of $m$ partitions with $|\lambda| = n$, and assume $f^\lambda(q) = \sum_k b_{\lambda,k} q^k$. Then for $k \in \mathbb{Z}$, $b_{\lambda,k} \neq 0$ if and only if

$$\frac{k - b(\alpha(\lambda))}{m} - b(\lambda) \in \left\{ 0, 1, \ldots, \left(\frac{n+1}{2}\right) - \sum_{c \in \lambda} h_c \right\} \setminus \mathcal{D}_\lambda,$$

where $\mathcal{D}_\lambda$ is empty unless $\lambda$ has a single non-empty partition $\lambda^{(i)}$ which is a rectangle with at least two rows and columns, in which case

$$\mathcal{D}_\lambda = \left\{ 1, \left(\frac{n+1}{2}\right) - \sum_{c \in \lambda^{(i)}} h_c - 1 \right\}.$$
Theorem

(Clifford Theory) The irreps of $G(m, d, n)$ are (more-or-less canonically) indexed by pairs $(\{\lambda\}^d, c)$ where $\lambda$ has $m$ parts and $n$ cells, $\{\lambda\}^d$ is its orbit under the size $d$ group of cyclic rotations, and $c$ is an element of the stabilizer of this orbit.
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(Stembridge ’89, BKS 18+)

$$f_{S\{\lambda\}, c}(q) = f_{\{\lambda\}^d}(q)$$

$$= \frac{\#\{\lambda\}^d}{d} \cdot \left[ \binom{n}{\alpha(\lambda)} \right]_{q;d} \cdot \prod_{i=1}^{m} f^{\lambda(i)}(q^m).$$
\(G(m, d, n)\) Internal Zeros

**(BKS 18+)** Let \(\lambda\) be a sequence of \(m\) partitions with \(|\lambda| = n \geq 1\), let \(d \mid m\), and let \(\{\lambda\}^d\) be the orbit of \(\lambda\) under the group \(C_d\) of \((m/d)\)-fold cyclic rotations. Then \(b_{\{\lambda\}^d,k \neq 0}\) if and only if for some \(\mu \in \{\lambda\}\) we have

\[
|\mu^{(1)}| + \cdots + |\mu^{(m/d)}| > 0 \text{ and }
\]

\[
\frac{k - b(\alpha(\mu))}{m} - b(\mu) \in \mathbb{Z}
\]

\[
\{0, 1, \ldots, |\mu^{(1)}| + \cdots + |\mu^{(m/d)}|\}
\]

\[
+ \binom{n}{2} - \sum_{c \in \mu} h_c \}
\]

\(\\mathcal{D}_{\mu;d}\).
Theorem

(Continued.) Here $\mathcal{D}_{\mu;d}$ is empty unless either

1. $\mu$ has a partition $\mu$ of size $n$; or

2. $\mu$ has a partition $\mu$ of size $n - 1$ and

$$|\mu^{(1)}| + \cdots + |\mu^{(m/d)}| = 1,$$

where in both cases $\mu$ must be a rectangle with at least two rows and columns. In case (1), we have

$$\mathcal{D}_{\mu;d} := \left\{ 1, \left( \frac{n + 1}{2} \right) - \sum_{c \in \mu} h_c - 1 \right\},$$

and in case (2) we have

$$\mathcal{D}_{\mu;d} := \left\{ 1, \left( \frac{n}{2} \right) - \sum_{c \in \mu} h_c \right\}.$$
Further work

- Probability: coefficients of $f_{\lambda}(q)$ are generally asymptotically normal. (To appear!)

- When does $\sum_{T \in \text{SYT}(\lambda/\mu)} q^{\text{maj}(T)}$ have internal zeros? (Mostly done; 5 extra block rules)

- Give a general, representation-theoretic interpretation of rotation rules

- The same for block rules?

- Unimodality classification conjecture

- Study deformed Gaussian binomial coefficients $\binom{n}{\alpha}_q$; $d$

- Conceptual explanation for primality corollary/why are rectangles special?
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Thanks!

THANKS!