INTRODUCTION TO $\lambda$-RINGS AND PLETHYSMS

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$\lambda$-RINGS AND PLETHYSMS

The purpose of this talk is to define and motivate the often-mysterious notion of “plethysm” of symmetric functions.

We begin with $\lambda$-rings. Abstractly, $\lambda$ rings are decategorifications of symmetric monoidal abelian categories. More concretely, let $R(G)$ be the representation ring over $\mathbb{C}$ of a finite group $G$, or more generally a compact Lie group, whose elements can be thought of as virtual characters. We have correspondences $+$ $\leftrightarrow$ $\oplus$ and $\cdot$ $\leftrightarrow$ $\otimes$. Given a $G$-module $V$, we can form a new $G$-module $\Lambda^k V$, the $k$th exterior power of $V$. On the level of the representation ring, this corresponds to the map (of sets!)

$$\lambda^k: R(G) \to R(G)$$

$$\lambda^k([V]) := [\Lambda^k V].$$

for $k \in \mathbb{Z}_{\geq 0}$.

Definition 1. A “non-special” $\lambda$-ring is a commutative, unital ring $R$ together with maps of sets $\lambda^k: R \to R$ for $k \in \mathbb{Z}_{\geq 0}$ such that for all $x, y \in R$

(i) $\lambda^0(x) = 1$
(ii) $\lambda^1(x) = x$
(iii) $\lambda^k(x + y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$.

(We’ll shortly remove the awkward phrase “non-special” by adding three more conditions.)

Example 2. The ring $R(G)$ satisfies (i) and (ii) trivially. Identity (iii) is a consequence of the natural isomorphism

$$\bigoplus_{i+j=k} (\Lambda^i U) \otimes (\Lambda^j V) \cong \Lambda^k (U \oplus V).$$

Example 3. When $G$ is the trivial group, $R(G) \cong \mathbb{Z}$ via $[V] \mapsto \dim V$. Hence we have a $\lambda$-ring structure on $\mathbb{Z}$ given by

$$\lambda^k(m) = \dim \Lambda^k (\mathbb{C}^m) = \binom{m}{k}.$$

Identity (iii) corresponds to the Vandermonde convolution identity

$$\binom{x + y}{k} = \sum_{i+j=k} \binom{x}{i} \binom{y}{j}.$$
Example 4. Let $G = \mathrm{GL}(\mathbb{C}^m)$ and let $R$ be the corresponding ring of polynomial representations. Let $\text{Sym}_m$ be the ring of symmetric polynomials in $m$ variables over $\mathbb{Z}$. We have the usual ring isomorphism

$$R \cong \text{Sym}_m,$$

$[V] \mapsto p_V(x_1, \ldots, x_m)$ where

$$p_V(x_1, \ldots, x_m) := \text{Tr}_V(\text{diag}(x_1, \ldots, x_m)).$$

For instance, $\mathbb{C}^m \mapsto x_1 + \cdots + x_m = e_1$. More generally,

$$\Lambda^k \mathbb{C}^m \mapsto e_k(x_1, \ldots, x_m)$$

since $\Lambda^k \mathbb{C}^m$ has a basis indexed by $k$-element subsets of any fixed basis of $\mathbb{C}^m$. The corresponding $\lambda$-ring structure on $\text{Sym}_m$, hence satisfies

$$(1) \quad \lambda^k(e_1) = e_k.$$

Example 5. Let $\text{Sym} := \lim_{m \to \infty} \text{Sym}_m$ be the usual ring of symmetric functions, which inherits maps $\lambda^k$ from the $\text{Sym}_m$ satisfying (1).

Question 6. Is $\lambda^k$ a ring homomorphism?

- Condition (iii) shows that while $\lambda^k$ is not (typically) additive, $\lambda^k(x + y)$ is determined by $\lambda^i(x), \lambda^j(y)$ for $i + j = k$.
- Similarly, while $\lambda^k$ is not (typically) multiplicative, when $R = \mathbb{Z}$ we may clearly write $\lambda^k(xy)$ in terms of $\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y)$. Similarly, $\lambda^h(\lambda^k(x))$ can be written in terms of $\lambda^1(x), \ldots, \lambda^{hk}(x)$.

Fact 7. There are “universal”, unique polynomials $P_k(x_1, \ldots, x_k, y_1, \ldots, y_k)$ and $P_{h,k}(x_1, \ldots, x_{hk})$ with integer coefficients such that in the rings $R(G)$, we have

$$[\Lambda^k(U \oplus V)] = P_k([\Lambda^1 U], \ldots, [\Lambda^k U], [\Lambda^1 V], \ldots, [\Lambda^k V])$$

and

$$[\Lambda^h(\Lambda^k U)] = P_{h,k}([\Lambda^1 U], \ldots, [\Lambda^{hk} U]).$$

Example 8. In $R(G)$, we have

$$\lambda^2(xy) = x^2(\lambda^2 x) + (\lambda^2 x)y^2 - 2(\lambda^2 x)(\lambda^2 y), \quad \lambda^2(\lambda^2 x) = x(\lambda^3 x) - \lambda^4 x.$$

For instance, the first of these corresponds to the (not-at-all-obvious) isomorphism

$$\Lambda^2(U \otimes V) \oplus 2(\Lambda^2 U \otimes \Lambda^2 V) \cong (U \otimes^2 \otimes \Lambda^2 U) \oplus (\Lambda^2 V \otimes V \otimes^2).$$

Definition 9. A $\lambda$-ring is a commutative, unital ring $R$ together with maps $\lambda^k : R \to R$ for $k \in \mathbb{Z}_{\geq 0}$ such that for all $x, y \in R$

(i) $\lambda^0(x) = 1$

(ii) $\lambda^1(x) = x$

(iii) $\lambda^k(x + y) = \sum_{i+j=k} \lambda^i(x)\lambda^j(y)$

(iv) $\lambda^k(xy) = P_k(\lambda^1(x), \ldots, \lambda^k(x), \lambda^1(y), \ldots, \lambda^k(y))$

(v) $\lambda^h(\lambda^k(x)) = P_{h,k}(\lambda^1(x), \ldots, \lambda^{hk}(x))$

(vi) $\lambda^k(1) = 0$ for $k > 1$.

Example 10. $R(G)$, $\text{Sym}_m$, and $\text{Sym}$ are all $\lambda$-rings.

The following fact connects us to plethysms. Note how checking that $e_1$ is a $\lambda$-ring generator of $\text{Sym}$ entails extensive use of (1) and conditions (iii)-(v).

Fact 11. $\text{Sym}$ is the free $\lambda$-ring on one generator.

A homomorphism of $\lambda$-rings is a ring morphism commuting with the $\lambda^k$ maps. The above fact can be rephrased as follows.

Proposition 12. Given a $\lambda$-ring $R$, for every $x \in R$ there is a unique homomorphism of $\lambda$-rings $\phi_x : \text{Sym} \to R$ such that $\phi_x(e_1) = x$. 
Definition 13. For \( g \in \text{Sym} \), let \( \phi_g : \text{Sym} \to \text{Sym} \) be the \( \lambda \)-ring homomorphism defined by \( \phi_g(e_1) = g \). For any \( f \in \text{Sym} \), the plethysm of \( g \) and \( f \) is
\[
f[g] := \phi_g(f).
\]
Equivalently, \( f[g] \) is can be defined by requiring
\[
\phi_f[g] = \phi_g \circ \phi_f.
\]
Eq. (2) says that plethysm of symmetric functions is (up to contravariance) precisely composition of \( \lambda \)-ring endomorphisms of \( \text{Sym} \). The explains the quite rare alternate notation \( g \circ f \) for \( f[g] \); a much more common alternate notation is \( f \circ g \), as discussed below. In any case, (2) shows that plethysm is associative in that
\[
f[g[h]] = (f[g])[h].
\]
Since \( \phi_{e_1} = \text{id} \), (2) also shows that \( e_1 \) is a two-sided identity for plethysm, namely that
\[
f[e_1] = e_1[f] = f.
\]
Example 14. We may now use (1) to identify the operators \( \lambda^k \) in terms of plethysms:
\[(3) \quad \lambda^k(g) = \lambda^k(\phi_g(e_1)) = \phi_g(\lambda^k(e_1)) = \phi_g(e_k) = e_k[g].\]

Example 15. For \( k \in \mathbb{Z}_{\geq 0} \), define maps \( \psi^k : R(G) \to R(G) \) by declaring that
\[
\chi_{\psi^k[V]}(g) = \chi[V](g^k)
\]
for all \( g \in G \) and all \( G \)-modules \( V \). It is straightforward to verify that \( \psi^k \) is a well-defined ring homomorphism. Likewise we have ring endomorphisms \( \psi^k : \text{Sym}_m \to \text{Sym}_m \). Since
\[
e_1(x_1, \ldots, x_m) = p_{C^n}(x_1, \ldots, x_m) = \text{Tr}_{C^n}(\text{diag}(x_1, \ldots, x_m)),
\]
we have
\[
\psi^k(e_1) = \text{Tr}_{C^n}(\text{diag}(x_1, \ldots, x_m)^k) = p_{C^n}(x_1^k, \ldots, x_m^k) = e_1(x_1^k, \ldots, x_m^k).
\]
The corresponding ring endomorphisms \( \psi^k \) on \( \text{Sym} \) thus satisfy
\[
\psi^k(e_1) = p_k.
\]
Fact 16. The ring endomorphisms \( \psi^k : \text{Sym} \to \text{Sym} \) just defined are \( \lambda \)-ring endomorphisms, i.e. \( \psi^k \circ \lambda^h = \lambda^h \circ \psi^k \).

Consequently, \( \psi^k = \phi_{\psi^k(e_1)} = \phi_{p_k} \). That is,
\[
\psi^k(g) = g[p_k].
\]
(In fact, \( p_k[g] = g[p_k] \).)

Remark 17. The operators \( \psi^k \) just defined are typically called Adams operations. They arise in \( K \)-theory, which involves an analogue of the ring \( R(G) \) where \( G \)-modules are replaced by vector bundles over a compact Hausdorff space. Indeed, Grothendieck originally defined \( \lambda \)-rings in the \( K \)-theory context.

We next give the first of two alternate definitions of plethysm.

Definition 18. Let \( \beta : \text{GL}(\mathbb{C}^p) \to \text{GL}(\mathbb{C}^n) \) and \( \alpha : \text{GL}(\mathbb{C}^n) \to \text{GL}(\mathbb{C}^m) \) be polynomial representations with characters
\[
g(x_1, \ldots, x_p) = \text{Tr}(\beta(\text{diag}(x_1, \ldots, x_p))) \in \text{Sym}_p
\]
\[
f(x_1, \ldots, x_n) = \text{Tr}(\alpha(\text{diag}(x_1, \ldots, x_n))) \in \text{Sym}_n.
\]
Then \( \alpha \beta : \text{GL}(\mathbb{C}^p) \to \text{GL}(\mathbb{C}^m) \) is a polynomial representation with character
\[(4) \quad f[g] := \text{Tr}(\alpha \beta(\text{diag}(x_1, \ldots, x_p))) \in \text{Sym}_p.
\]

Remark 19. Definition 18 explains the relatively common alternate notation \( f \circ g \) for \( f[g] \). To connect Definition 13 and Definition 18, fix \( g \in \text{Sym}_p \) and consider the map \( \text{Sym}_n \to \text{Sym}_p \) given by \( f \mapsto f[g] \) using (4). This is easily seen to be a ring homomorphism, and in fact it commutes with the \( \lambda^k \), so is a \( \lambda \)-ring homomorphism. Finish off by taking limits and computing \( e_1 \mapsto g \).
Definition 20. The following is the usual combinatorial definition of plethysm. Write \( g \in \text{Sym} \) as \( g = \sum_{u \in I} u \) where \( I \) is a multiset of monomials. Then for any \( f \in \text{Sym} \),
\[
f[g] := f(u : u \in I).
\]

Remark 21. To connect Definition 13 and Definition 20, as before, note that \( f \mapsto f(u : u \in I) \) is a well-defined ring endomorphism of \( \text{Sym} \). One must check that it commutes with the \( \lambda^k \) maps. Since we clearly have \( e_1 \mapsto \sum_{u \in I} u = g \), the equivalence of the definitions follows.

Remark 22. Further sources:

- [ncatlab.org/nlab/show/Lambda-ring](https://ncatlab.org/nlab/show/Lambda-ring) for abstract overview of \( \lambda \)-rings
- [http://www-math.mit.edu/~rstan/transparencies/plethysm.pdf](http://www-math.mit.edu/~rstan/transparencies/plethysm.pdf) for some slides from Richard Stanley discussing Definition 18 and Definition 20 above
- [http://www.math.jhu.edu/~jmb/note/adamrept.pdf](http://www.math.jhu.edu/~jmb/note/adamrept.pdf) for a lovely little note on Adams operations for representations of compact Lie groups
- [http://www.cip.ifi.lmu.de/~grinberg/algebra/lambda.pdf](http://www.cip.ifi.lmu.de/~grinberg/algebra/lambda.pdf) for many gory details on \( \lambda \)-rings, e.g. constructing \( P_k \) and \( P_{h,k} \); highly unpolished and incomplete
- [Ful97][Part II] for a nice summary of representation theory over \( GL(\mathbb{C}^n) \)
- [Mac95][I.8 and I.A] for Macdonald’s concise account of plethysm and its connection to symmetric group representation theory
- [Knu73] for a book on \( \lambda \) rings and symmetric functions; somewhat old, apparently not the best regarded, I didn’t actually take the time to acquire a copy; written by Donald Knutson, apparently father of Allen Knutson
- [Yau10] for a more modern book on \( \lambda \) rings, though I didn’t actually take the time to acquire a copy

References


