

COUNTING DESCENT PAIRS WITH PRESCRIBED TOPS AND BOTTOMS

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ABSTRACT. Given sets X and Y of positive integers and a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, an X, Y -descent of σ is a descent pair $\sigma_i > \sigma_{i+1}$ whose “top” σ_i is in X and whose “bottom” σ_{i+1} is in Y . We give two formulas for the number $P_{n,s}^{X,Y}$ of $\sigma \in S_n$ with s X, Y -descents. $P_{n,s}^{X,Y}$ is also shown to be a hit number of a certain Ferrers board. This work generalizes results of Kitaev and Remmel [15] [16] on counting descent pairs whose top (or bottom) is equal to $0 \pmod k$.

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1. INTRODUCTION

Let S_n denote the set of permutations of the set $[n] = \{1, 2, \dots, n\}$. A *descent pair* of a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$ is a pair (σ_i, σ_{i+1}) with $\sigma_i > \sigma_{i+1}$. The main focus of this paper is to study the distribution of descent pairs whose top σ_i lies in some fixed set X and whose bottom σ_{i+1} lies in some fixed set Y .

Definition 1.1. *Given subsets $X, Y \subseteq \mathbb{N}$ and a permutation $\sigma \in S_n$, let*

$$\begin{aligned} Des_{X,Y}(\sigma) &= \{i : \sigma_i > \sigma_{i+1} \text{ \& } \sigma_i \in X \text{ \& } \sigma_{i+1} \in Y\}, \text{ and} \\ des_{X,Y}(\sigma) &= |Des_{X,Y}(\sigma)|. \end{aligned}$$

If $i \in Des_{X,Y}(\sigma)$, then we call the pair (σ_i, σ_{i+1}) an X, Y -descent.

For example, if $X = \{2, 3, 5\}, Y = \{1, 3, 4\}$, and $\sigma = 54213$, then $Des_{X,Y}(\sigma) = \{1, 3\}$ and $des_{X,Y}(\sigma) = 2$.

For fixed n we define the polynomial

$$P_n^{X,Y}(x) = \sum_{s \geq 0} P_{n,s}^{X,Y} x^s := \sum_{\sigma \in S_n} x^{des_{X,Y}(\sigma)}. \tag{1.1}$$

Thus the coefficient $P_{n,s}^{X,Y}$ is the number of $\sigma \in S_n$ with exactly s X, Y -descents.

Our main result is to give direct combinatorial proofs of a pair of formulas for $P_{n,s}^{X,Y}$. First of all, for any set $S \subseteq \mathbb{N}$, let

$$\begin{aligned} S_n &= S \cap [n], \text{ and} \\ S_n^c &= (S^c)_n = [n] - S. \end{aligned}$$

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Then we have

Theorem 2.3

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^s (-1)^{s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1+r + \alpha_{X,n,x} + \beta_{Y,n,x}), \quad (1.2)$$

and

Theorem 2.5

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{|X_n|-s-r} \prod_{x \in X_n} (r + \beta_{X,n,x} - \beta_{Y,n,x}), \quad (1.3)$$

where for any set S and any $j, 1 \leq j \leq n$, we define

$$\begin{aligned} \alpha_{S,n,j} &= |S^c \cap \{j+1, j+2, \dots, n\}| = |\{x : j < x \leq n \text{ \& } x \notin S\}|, \text{ and} \\ \beta_{S,n,j} &= |S^c \cap \{1, 2, \dots, j-1\}| = |\{x : 1 \leq x < j \text{ \& } x \notin S\}|. \end{aligned}$$

Example 1.2. Suppose $X = \{2, 3, 4, 6, 7, 9\}, Y = \{1, 4, 8\}$, and $n = 6$. Thus $X_6 = \{2, 3, 4, 6\}, X_6^c = \{1, 5\}, Y_6 = \{1, 4\}, Y_6^c = \{2, 3, 5, 6\}$, and we have the following table of values of $\alpha_{X,6,x}, \beta_{Y,6,x}$, and $\beta_{X,6,x}$.

x	2	3	4	6
$\alpha_{X,6,x}$	1	1	1	0
$\beta_{Y,6,x}$	0	1	2	2
$\beta_{X,6,x}$	1	1	1	2

(1.2) gives

$$\begin{aligned} P_{6,2}^{X,Y} &= 2! \sum_{r=0}^2 (-1)^{2-r} \binom{2+r}{r} \binom{7}{2-r} (2+r)(3+r)(4+r)(3+r) \\ &= 2(1 \cdot 21 \cdot 2 \cdot 3 \cdot 4 \cdot 3 - 3 \cdot 7 \cdot 3 \cdot 4 \cdot 5 \cdot 4 + 6 \cdot 1 \cdot 4 \cdot 5 \cdot 6 \cdot 5) \\ &= 2(1512 - 5040 + 3600) \\ &= 144. \end{aligned}$$

while (1.3) gives

$$\begin{aligned} P_{6,2}^{X,Y} &= 2! \sum_{r=0}^2 (-1)^{2-r} \binom{2+r}{r} \binom{7}{2-r} (1+r)(0+r)(-1+r)(0+r) \\ &= 2(1 \cdot 21 \cdot 1 \cdot 0 \cdot (-1) \cdot 0 - 3 \cdot 7 \cdot 2 \cdot 1 \cdot 0 \cdot 1 + 6 \cdot 1 \cdot 3 \cdot 2 \cdot 1 \cdot 2) \\ &= 2(0 - 0 + 72) \\ &= 144. \end{aligned}$$

Let $P_{n,s}^X := P_{n,s}^{X,\mathbb{N}}$. Since $\beta_{\mathbb{N},n,x} = 0$ for all $x \in X_n$, we have as corollaries

$$P_{n,s}^X = |X_n^c|! \sum_{r=0}^s (-1)^{s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1+r + \alpha_{X,n,x}) \quad (1.4)$$

and

$$P_{n,s}^X = |X_n^c|! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{|X_n|-s-r} \prod_{x \in X_n} (r + \beta_{X,n,x}). \quad (1.5)$$

We will show that the equality of (1.4) and (1.5) is equivalent to a certain special case of a general transformation result due to Gasper [6] for hypergeometric series of Karlsson-Minton type. Since (1.4) and (1.5) have completely combinatorial proofs, it follows that we can give combinatorial proofs of many special cases of Gasper's transformation theorem.

We will use the fundamental transformation of Foata [4] to show that the polynomials $P_n^{X,Y}(x)$ are special cases of hit polynomials for Ferrers boards B contained in the $n \times n$ board. Since there are many formulas for hit polynomials of Ferrers boards (see, for example, the results of Haglund

[9]), we have an alternative way to prove formulas for the polynomials $P_n^{X,Y}(x)$. Moreover, we will show that we can use the same idea to reduce the computation of the coefficients of $P_n^{X,Y}(x)$ to the computation of the coefficients of $P_n^{X^*,\mathbb{N}}(x)$, for some appropriate X^* depending on X and Y . Thus (1.4) and (1.5) already contain all of the information needed to compute the seemingly more general formulas (1.2) and (1.3).

This type of study was initiated by Kitaev and Remmel [15, 16]. In particular, they studied descents according to the equivalence class mod k of either the top or bottom of a descent pair. For any set $X \subseteq \{1, 2, 3, \dots\}$, they defined

$$\begin{aligned} & \overleftarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \text{ \& } \sigma_i \in X\} \text{ and } \overleftarrow{des}_X(\sigma) = |\overleftarrow{Des}_X(\sigma)|, \text{ and} \\ & \overrightarrow{Des}_X(\sigma) = \{i : \sigma_i > \sigma_{i+1} \text{ \& } \sigma_{i+1} \in X\} \text{ and } \overrightarrow{des}_X(\sigma) = |\overrightarrow{Des}_X(\sigma)|. \end{aligned}$$

It is easy to see that $\overleftarrow{Des}_X(\sigma) = Des_{X,\mathbb{N}}(\sigma)$ and $\overrightarrow{Des}_X(\sigma) = Des_{\mathbb{N},X}(\sigma)$. In [15], Kitaev and Remmel studied polynomials such as

$$\begin{aligned} R_n(x) &= \sum_{k \geq 0} R_{k,n} x^k := \sum_{\sigma \in S_n} x^{\overleftarrow{des}_E(\sigma)}, \text{ and} \\ Q_n(x) &= \sum_{k \geq 0} Q_{k,n} x^k := \sum_{\sigma \in S_n} x^{\overrightarrow{des}_E(\sigma)}, \end{aligned}$$

where $E = \{2, 4, 6, \dots\}$ is the set of positive even integers. In these cases, they found surprisingly simple formulas for the coefficients. For example, they showed that

$$R_{2n,k} = \binom{n}{k}^2 (n!)^2. \quad (1.6)$$

In [16], Kitaev and Remmel studied polynomials such as

$$\begin{aligned} A_n^{(k)}(x) &= \sum_{j \geq 0} A_{n,j}^{(k)} x^j := \sum_{\sigma \in S_n} x^{\overleftarrow{des}_{k\mathbb{N}}(\sigma)}, \text{ and} \\ B_n^{(k)}(x) &= \sum_{j \geq 0} B_{n,j}^{(k)} x^j := \sum_{\sigma \in S_n} x^{\overrightarrow{des}_{k\mathbb{N}}(\sigma)}, \end{aligned}$$

where $k\mathbb{N} = \{k, 2k, 3k, \dots\}$. Note that $A_n^{(k)}(x) = P_n^{X,\mathbb{N}}(x)$ and $B_n^{(k)}(x) = P_n^{\mathbb{N},Y}(x)$, where X and Y are $k\mathbb{N}$ for some $k \geq 2$. When $k \geq 3$, the formulas for the coefficients of these polynomials are not as simple as (1.6). For example, Kitaev and Remmel used a recursion to prove the following formulas, which hold for all $0 \leq j \leq k-1$ and all $n \geq 0$.

$$\begin{aligned} A_{kn+j,s}^{(k)} &= ((k-1)n+j)! \times \\ & \sum_{r=0}^s (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=0}^{n-1} (r+1+j+(k-1)i) \quad (1.7) \end{aligned}$$

$$\begin{aligned} A_{kn+j,s}^{(k)} &= ((k-1)n+j)! \times \\ & \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{n-s-r} \prod_{i=1}^n (r+(k-1)i) \quad (1.8) \end{aligned}$$

Our main results are generalizations of the two formulas (1.7) and (1.8).

The outline of this paper is as follows. In Section 2, we give several formulations of a recursion for the number $P_{n,s}^{X,Y}$ of $\sigma \in S_n$ with s X, Y -descents. Then we present our main results, the combinatorial proofs of (1.2) and (1.3). In Section 3, we give several applications of our main results, including new proofs of results of Kitaev and Remmel [15] [16] on counting descent pairs whose top (or bottom) is equivalent to $0 \pmod k$, and a combinatorial proof of various special cases of a transformation of Karlsson-Minton type hypergeometric series due to Gasper [6]. In Section 4, we use Foata's First Transformation, which is a bijection taking descents to excedences, to rephrase the problem of computing the polynomials $P_n^{X,Y}(x)$ as one of computing hit polynomials for certain

boards B contained in the $n \times n$ board. In Section 5, we show that our results can be extended from permutations to words. In Section 6, we shall consider a more general problem. That is, for any $X, Y, Z \subseteq \mathbb{N}$, we can consider the polynomials

$$P_n^{X,Y,Z}(x) = \sum_{s \geq 0} P_{n,s}^{X,Y,Z} x^s := \sum_{\sigma \in S_n} x^{\text{des}_{X,Y,Z}(\sigma)},$$

where for any subsets X, Y , and Z of \mathbb{N} , and permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$,

$$\begin{aligned} \text{Des}_{X,Y,Z}(\sigma) &= \{i : \sigma_i > \sigma_{i+1} \text{ \& } \sigma_i \in X, \sigma_{i+1} \in Y \text{ \& } \sigma_i - \sigma_{i+1} \in Z\}, \text{ and} \\ \text{des}_{X,Y,Z}(\sigma) &= |\text{Des}_{X,Y,Z}(\sigma)|. \end{aligned}$$

Clearly $P_n^{X,Y}(x) = P_n^{X,Y,\mathbb{N}}(x)$. We do not have a formula for the coefficients $P_{n,s}^{X,Y,Z}$ for arbitrary X, Y , and Z . However, we will show that we can find formulas for $P_{n,s}^{X,Y,Z}$ for certain special cases of X, Y , and Z . Finally, in Section 7, we present some directions for future research.

2. PRESCRIBED TOPS AND BOTTOMS

In this section, we will give several ways to compute the coefficients $P_{n,s}^{X,Y}$.

Given $X, Y \subseteq \mathbb{N}$, let $P_0^{X,Y}(x, y) = 1$, and for $n \geq 1$, define

$$P_n^{X,Y}(x, y) = \sum_{s,t \geq 0} P_{n,s,t}^{X,Y} x^s y^t := \sum_{\sigma \in S_n} x^{\text{des}_{X,Y}(\sigma)} y^{|\mathbb{N}_n^c|}.$$

Let Φ_{n+1} and Ψ_{n+1} be the operators defined as

$$\begin{aligned} \Phi_{n+1} : x^s y^t &\longrightarrow s x^{s-1} y^t + (n+1-s) x^s y^t \\ \Psi_{n+1} : x^s y^t &\longrightarrow (s+t+1) x^s y^t + (n-s-t) x^{s+1} y^t. \end{aligned}$$

Proposition 2.1. *For any sets $X, Y \subseteq \mathbb{N}$, the polynomials $P_n^{X,Y}(x, y)$ satisfy*

$$P_{n+1}^{X,Y}(x, y) = \begin{cases} y \cdot \Phi_{n+1}(P_n^{X,Y}(x, y)) & \text{if } n+1 \notin X \text{ and } n+1 \notin Y, \\ \Phi_{n+1}(P_n^{X,Y}(x, y)) & \text{if } n+1 \notin X \text{ and } n+1 \in Y, \\ y \cdot \Psi_{n+1}(P_n^{X,Y}(x, y)) & \text{if } n+1 \in X \text{ and } n+1 \notin Y, \text{ and} \\ \Psi_{n+1}(P_n^{X,Y}(x, y)) & \text{if } n+1 \in X \text{ and } n+1 \in Y. \end{cases}$$

Proof. We think of a permutation as being built up by successively inserting the numbers 1, 2, 3, and so on. Given a permutation $\sigma \in S_n$ with s X, Y -descents, if $n+1 \notin X$ and we insert $n+1$ in the middle of one of the s X, Y -descent pairs, then we destroy that X, Y -descent and get a permutation with $s-1$ X, Y -descents. If, instead, we insert $n+1$ in one of the other $n+1-s$ possible spots (including the spots at the beginning and end of the permutation), then we preserve the number of X, Y -descents. Thus, if $n+1 \notin X$, we have

$$P_{n+1}^{X,Y}(x, y) = \begin{cases} y \cdot \Phi_{n+1}(P_n^{X,Y}(x, y)) & \text{if } n+1 \notin Y, \text{ and} \\ \Phi_{n+1}(P_n^{X,Y}(x, y)) & \text{if } n+1 \in Y. \end{cases}$$

On the other hand, if $n+1 \in X$, then we preserve the number of X, Y -descents by inserting $n+1$ in the middle of one of the s X, Y -descent pairs, or before any of the t elements of \mathbb{N}_n^c , or at the end of the permutation. If, instead, we insert $n+1$ in one of the other $n-s-t$ possible spots we create a new X, Y -descent. Thus, if $n+1 \in X$, we have

$$P_{n+1}^{X,Y}(x, y) = \begin{cases} y \cdot \Psi_{n+1}(P_n^{X,Y}(x, y)) & \text{if } n+1 \notin Y, \text{ and} \\ \Psi_{n+1}(P_n^{X,Y}(x, y)) & \text{if } n+1 \in Y. \end{cases}$$

□

It is easy to see that Proposition 2.1 implies the following result.

Corollary 2.2. *For all $X, Y \subseteq \mathbb{N}$ and $n \geq 1$, the following recursion holds for the coefficients $P_{n,s,t}^{X,Y}$.*

$$P_{n+1,s,t}^{X,Y} = \begin{cases} (s+1)P_{n,s+1,t-1}^{X,Y} + (n+1-s)P_{n,s,t-1}^{X,Y} & \text{if } n+1 \notin X \text{ and } n+1 \notin Y, \\ (s+1)P_{n,s+1,t}^{X,Y} + (n+1-s)P_{n,s,t}^{X,Y} & \text{if } n+1 \notin X \text{ and } n+1 \in Y, \\ (s+t)P_{n,s,t-1}^{X,Y} + (n+2-s-t)P_{n,s-1,t-1}^{X,Y} & \text{if } n+1 \in X \text{ and } n+1 \notin Y, \text{ and} \\ (s+t+1)P_{n,s,t}^{X,Y} + (n+1-s-t)P_{n,s-1,t}^{X,Y} & \text{if } n+1 \in X \text{ and } n+1 \in Y. \end{cases} \quad (2.1)$$

We can rephrase Proposition 2.1 in terms of partial differential equations as follows.

$$P_{n+1}^{X,Y}(x,y) = \begin{cases} y((n+1)P_n^{X,Y}(x,y) + (1-x)\frac{\partial}{\partial x}P_n^{X,Y}(x,y)) & \text{if } n+1 \notin X \text{ and } n+1 \notin Y, \\ (n+1)P_n^{X,Y}(x,y) + (1-x)\frac{\partial}{\partial x}P_n^{X,Y}(x,y) & \text{if } n+1 \notin X \text{ and } n+1 \in Y, \\ y((xn+1)P_n^{X,Y}(x,y) + (x-x^2)\frac{\partial}{\partial x}P_n^{X,Y}(x,y) + y(1-x)\frac{\partial}{\partial y}P_n^{X,Y}(x,y)) & \text{if } n+1 \in X \text{ and } n+1 \notin Y, \text{ and} \\ (xn+1)P_n^{X,Y}(x,y) + (x-x^2)\frac{\partial}{\partial x}P_n^{X,Y}(x,y) + y(1-x)\frac{\partial}{\partial y}P_n^{X,Y}(x,y) & \text{if } n+1 \in X \text{ and } n+1 \in Y. \end{cases} \quad (2.2)$$

One can then use either of the recursions (2.1) and (2.2) to compute $P_n^{X,Y}(x,y)$ for any X and Y . For example, if $X = \{2, 3, 5\}$ and $Y = \{1, 3, 4\}$ we have

$$\begin{aligned} P_0^{X,Y}(x,y) &= 1 \\ P_1^{X,Y}(x,y) &= 1 \\ P_2^{X,Y}(x,y) &= y(1+x) \\ P_3^{X,Y}(x,y) &= y(2+4x) \\ P_4^{X,Y}(x,y) &= y(12+12x) \\ P_5^{X,Y}(x,y) &= y^2(24+72x+24x^2). \end{aligned}$$

It is easy to see that $P_{n,s,t}^{X,Y} = 0$ unless $t = |Y_n^c|$, so we will drop the t and write $P_{n,s}^{X,Y}$ for $P_{n,s,|Y_n^c|}^{X,Y}$. Thus $P_n^{X,Y}(x,1)$ is really just the polynomial $P_n^{X,Y}(x)$ defined by (1.1).

Recall that for each $x \in X_n$, we defined

$$\begin{aligned} \alpha_{X,n,x} &= |\{z : x < z \leq n \text{ \& } z \notin X\}|, \text{ and} \\ \beta_{Y,n,x} &= |\{z : 1 \leq z < x \text{ \& } z \notin Y\}|. \end{aligned}$$

Then we have following formula for $P_{n,s}^{X,Y}$.

Theorem 2.3.

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^s (-1)^{s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1+r+\alpha_{X,n,x}+\beta_{Y,n,x}). \quad (2.3)$$

Remark 2.4. *Theorem 2.3 can be proved by showing that the formula satisfies the recursion (2.1) for $P_{n,s}^{X,Y}$. However, we will give a direct combinatorial proof using a sign-reversing involution on a set of configurations, which are arrays of numbers, +’s, and –’s. The basic idea is simple: applying the involution to each configuration results in either changing a + to a –, or changing a – to a +. The fixed points of the involution will be shown to correspond naturally to permutations $\sigma \in S_n$ such that $\text{des}_{X,Y}(\sigma) = s$.*

Proof. Let X, Y, n , and s be given. For r satisfying $0 \leq r \leq s$, we define the set of what we call $(n, s, r)^{X,Y}$ -configurations. An $(n, s, r)^{X,Y}$ -configuration c consists of an array of the numbers $1, 2, \dots, n, r$ +’s, and $(s-r)$ –’s, satisfying the following two conditions:

- (i) each – is either at the very beginning of the array or immediately follows a number, and
- (ii) if $x \in X$ and $y \in Y$ are consecutive numbers in the array, and $x > y$, i.e., if (x, y) forms an X, Y -descent pair in the underlying permutation, then there must be at least one + between x and y .

Note that in an $(n, s, r)^{X,Y}$ -configuration, the number of +’s plus the number of –’s equals s . For example, if $X = \{2, 3, 5, 6\}$ and $Y = \{1, 3\}$, the following is a $(6, 5, 3)^{X,Y}$ -configuration.

$$c = 5 + 2 - +46 + 13-$$

In this example, the underlying permutation is 524613. In general, we will let $c_1 c_2 \cdots c_n$ denote the underlying permutation of the $(n, s, r)^{X, Y}$ -configuration c .

Let $C_{n, s, r}^{X, Y}$ be the set of all $(n, s, r)^{X, Y}$ -configurations. We claim that

$$|C_{n, s, r}^{X, Y}| = |X_n^c|! \binom{|X_n^c| + r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1 + r + \alpha_{X, n, x} + \beta_{Y, n, x}).$$

That is, we can construct the $(n, s, r)^{X, Y}$ -configurations as follows. First, we pick an order for the elements in X_n^c . This can be done in $|X_n^c|!$ ways. Next, we insert the r +’s. This can be done in $\binom{|X_n^c| + r}{r}$ ways. Next, we insert the elements of $X_n = \{x_1 < x_2 < \cdots < x_{|X_n|}\}$ in increasing order. After placing x_1, x_2, \dots, x_{i-1} , the next element x_i can go

- immediately before any of the β_{Y, n, x_i} elements of $\{1, 2, \dots, x_{i-1}\}$ that is not in Y , or
- immediately before any of the α_{X, n, x_i} elements of $\{x_i + 1, x_i + 2, \dots, n\}$ that is not in X , or
- immediately before any of the r +’s, or
- at the very end of the array.

Thus we can place the elements of X_n in $\prod_{i=1}^{|X_n|} (1 + r + \alpha_{X, n, x} + \beta_{Y, n, x})$ ways. Note that although x_i might also be in Y , and might be placed immediately *after* some other element of X_n , condition (ii) is not violated because the elements of X_n are placed in increasing order. Finally, since each $-$ must occur either at the very start of the configuration or immediately following a number, we can place the $-$ ’s in $\binom{n+1}{s-r}$ ways.

We define the weight $w(c)$ of an $(n, s, r)^{X, Y}$ -configuration c to be $(-1)^{s-r}$, i.e., -1 to the number of $-$ ’s of c . It then follows that the RHS of (2.3) equals

$$\sum_{r=0}^s \sum_{c \in C_{n, s, r}^{X, Y}} w(c).$$

We now prove the theorem by exhibiting a sign-reversing involution I on the set $C_{n, s}^{X, Y} = \bigsqcup_{r=0}^s C_{n, s, r}^{X, Y}$, whose fixed points correspond to permutations $\sigma \in S_n$ such that $\text{des}_{X, Y}(\sigma) = s$. We say that a sign can be “reversed” if it can be changed from $+$ to $-$ or from $-$ to $+$ without violating conditions (i) and (ii). To apply I to a configuration c , we scan from left to right until we find the first sign that can be reversed. We then reverse that sign, and we let $I(c)$ be the resulting configuration. If no signs can be reversed, we set $I(c) = c$.

In the example above, the first sign we encounter is the $+$ following 5. This $+$ can be reversed, since 52 is not an X, Y -descent. Thus $I(c)$ is the configuration shown below.

$$I(c) = 5 - 2 - +46 + 13 -$$

It is easy to see that $I(I(c)) = c$ in this case, since applying I again we change the $-$ following 5 back to a $+$.

As another example, suppose $X = E, Y = O$, and $n = 9$. Let c be the following $(9, 4, 3)^{X, Y}$ -configuration.

$$c = 986 + 17 - + + 4253$$

In this example we cannot reverse the $+$ following 6, because 61 is an X, Y -descent in the underlying permutation 986174253. Thus we move on to the $-$ following 7. Changing this $-$ to a $+$ we get

$$I(c) = 986 + 17 + + + 4253.$$

It is easy to see that $I(I(c)) = c$ in this case, as well.

Conditions (i) and (ii) are clearly preserved by the very definition of I . It is also clear that I is sign-reversing, since if $I(c) \neq c$, then $I(c)$ either has one more $-$ than c , or one fewer $-$ than c . To see that I is in fact an involution, we note that the only signs that are *not* reversible are single $+$ ’s occurring in the middle of an X, Y -descent pair, and $+$ ’s that immediately follows another sign. In either case, it is clear that a sign is reversible in a configuration c if and only if the corresponding sign is reversible in $I(c)$. Thus, if a sign is the first reversible sign in c , the corresponding sign in

$I(c)$ must also be the first reversible sign in $I(c)$. It follows that $I(I(c)) = c$ for all $c \in C_{n,s}^{X,Y}$. We therefore have

$$\sum_{r=0}^s \sum_{c \in C_{n,s,r}^{X,Y}} w(c) = \sum_{r=0}^s \sum_{\substack{c \in C_{n,s,r}^{X,Y} \\ I(c) = c}} w(c).$$

Now, consider the fixed points of I . Suppose that $I(c) = c$. Then c clearly can have no $-$'s, and so $r = s$ and $w(c) = 1$. It must also be the case that no $+$'s can be reversed. Thus each of the s $+$'s must occur singly in the middle of an X, Y -descent pair. It follows that the underlying permutation has exactly s X, Y -descents.

Finally, we should observe that if $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n$ is a permutation with exactly s X, Y -descents, then we can create a fixed point of I simply by placing a $+$ in the middle of each X, Y -descent pair. For example, if $X = \{2, 4, 6, 9\}$, $Y = \{1, 4, 7\}$, $n = 9$, $s = 2$, and $\sigma = 528941637$, then we have

$$c = 5289 + 4 + 1637.$$

□

Note that the right-hand side of (2.3) makes sense for $s > |X_n|$, even though a permutation $\sigma \in S_n$ can have no more than $|X_n|$ X, Y -descents. This issue becomes important if one attempts to prove (2.3) by induction, using the recursion (2.1). Although it is straightforward to show that the right-hand side of (2.3) satisfies the recursion, to complete the proof one needs to show independently that the sum is zero when $s = |X_n| + 1$. Our involution makes it clear that the sum is zero for *all* $s > |X_n|$, since in such cases there will always be at least one $-$ or at least $(|X_n| + 1)$ $+$'s. If there are at least $(|X_n| + 1)$ $+$'s, then there must be either an X, Y -descent (c_i, c_{i+1}) such that at least two $+$'s occur between c_i and c_{i+1} , or consecutive numbers c_i and c_{i+1} which do not form X, Y -descent, such that at least one $+$ appears between c_i and c_{i+1} . In each of these cases we can change a sign, and thus $I(c) \neq c$ for all $c \in C_{n,s}^{X,Y}$.

We can use the same involution on a related set of objects to prove an alternative formula for $P_{n,s}^{X,Y}$.

Theorem 2.5.

$$P_{n,s}^{X,Y} = |X_n^c|! \sum_{r=0}^{|X_n|-s} (-1)^{|X_n|-s-r} \binom{|X_n^c|+r}{r} \binom{n+1}{|X_n|-s-r} \prod_{x \in X_n} (r + \beta_{X,n,x} - \beta_{Y,n,x}). \quad (2.4)$$

Proof. Let X, Y, n , and s be given. For r satisfying $0 \leq r \leq |X_n| - s$, an $\overline{(n, s, r)}^{X,Y}$ -configuration consists of an array of the numbers $1, 2, \dots, n, r$ $+$'s, and $(|X_n| - s - r)$ $-$'s, satisfying the following three conditions.

- (i) each $-$ is either at the very beginning of the array or immediately follows a number,
- (ii) if $c_i \in X$, $1 \leq i < n$, and (c_i, c_{i+1}) is not an X, Y -descent pair of the underlying permutation, then there must be at least one $+$ between c_i and c_{i+1} , and
- (iii) if $c_n \in X$, then at least one $+$ must occur to the right of c_n .

Note that in an $\overline{(n, s, r)}^{X,Y}$ -configuration, the number of $+$'s plus the number of $-$'s equals $|X_n| - s$.

As an example, if $X = \{2, 3, 6\}$ and $Y = \{1, 2, 5\}$, then the following is a $\overline{(6, 1, 1)}^{X,Y}$ -configuration.

$$c = 213 + 6 - 54$$

Let $\overline{C}_{n,s,r}^{X,Y}$ be the set of all $\overline{(n, s, r)}^{X,Y}$ -configurations. Then we claim that

$$|\overline{C}_{n,s,r}^{X,Y}| = |X_n^c|! \binom{|X_n^c|+r}{r} \binom{n+1}{|X_n|-s-r} \prod_{x \in X_n} (r + \beta_{X,n,x} - \beta_{Y,n,x}).$$

That is, we can construct the $\overline{(n, s, r)}^{X,Y}$ -configurations as follows. First, we pick an order for the elements in X_n^c . This can be done in $|X_n^c|!$ ways. Next, we insert the r $+$'s. This can be done in $\binom{|X_n^c|+r}{r}$ ways. Next, we insert the elements of $X_n = \{x_1 < x_2 < \cdots < x_{|X_n|}\}$ in increasing order. First, we can place x_1 in $r + \beta_{X,n,x_1} - \beta_{Y,n,x_1}$ ways, since x_1 can either go immediately before any of

the r +’s or immediately before any of the $x_1 - 1 - \beta_{Y,n,x_1} = \beta_{X,n,x_1} - \beta_{Y,n,x_1}$ elements of Y which are less than x_1 . We note here that $\beta_{X,n,x_i} = x_i - i$ for all $i, 1 \leq i \leq |X_n|$. There are now two cases for placing x_2 .

Case 1. x_1 was placed immediately in front of some element of $y \in Y$. In this case, x_2 cannot be placed immediately in front of y , since otherwise we would violate condition (ii). x_2 can be placed before any + or immediately in front of any element of Y which is less than x_2 , except y . Hence, x_2 can be placed in

$$\begin{aligned} r + x_2 - 1 - \beta_{Y,n,x_2} - 1 &= r + x_2 - 2 - \beta_{Y,n,x_2} \\ &= r + \beta_{X,n,x_2} - \beta_{Y,n,x_2} \end{aligned}$$

ways.

Case 2. x_1 was placed immediately before a +. In this case, x_2 cannot be placed immediately before the same +, since again we would violate condition (ii). x_2 can be placed immediately before any of the other +’s or immediately before any element of Y which is less than x_2 . Hence x_2 can be placed in

$$\begin{aligned} r - 1 + x_2 - 1 - \beta_{Y,n,x_2} &= r + x_2 - 2 - \beta_{Y,n,x_2} \\ &= r + \beta_{X,n,x_2} - \beta_{Y,n,x_2} \end{aligned}$$

ways.

In general, having placed x_1, x_2, \dots, x_{i-1} , we cannot place x_i immediately before some $y \in Y, y < x_i$, which earlier had an element of $\{x_1, x_2, \dots, x_{i-1}\}$ placed before it. Similarly, we cannot place x_i immediately before any + which earlier had an element of $\{x_1, x_2, \dots, x_{i-1}\}$ placed before it. It then follows that there are

$$\begin{aligned} r + x_i - 1 - \beta_{Y,n,x_i} - (i - 1) &= r + x_i - i - \beta_{Y,n,x_i} \\ &= r + \beta_{X,n,x_i} - \beta_{Y,n,x_i} \end{aligned}$$

ways to place x_i . Thus, there are total of $\prod_{i=1}^{|X_n|} (r + \beta_{X,n,x_i} - \beta_{Y,n,x_i})$ ways to place $x_1, x_2, \dots, x_{|X_n|}$, given our placement of the elements of X_n^c . Finally, we can place the $-$ ’s in $\binom{n+1}{|X_n| - s - r}$ ways.

We define the weight $w(c)$ of an $(\overline{n, s, r})^{X, Y}$ -configuration c to be $(-1)^{|X_n| - s - r}$, i.e., -1 to the number of $-$ ’s of c . It then follows that the RHS of (2.4) equals

$$\sum_{r=0}^{|X_n| - s} \sum_{c \in \overline{C}_{n, s, r}^{X, Y}} w(c),$$

We now prove the theorem by exhibiting a sign-reversing involution I on the set $\overline{C}_{n, s}^{X, Y} = \bigsqcup_{r=0}^{|X_n| - s} \overline{C}_{n, s, r}^{X, Y}$ whose fixed points correspond to permutations $\sigma \in S_n$ such that $des_{X, Y}(\sigma) = s$. We define I exactly as in the proof of Theorem 2.3. That is, we scan from left to right and reverse the first sign that we can reverse without violating conditions (i)-(iii).

In the example above, we cannot reverse the + following 3 without violating condition (ii), since $3 \in X$ and 36 is not an X, Y -descent. Thus, we reverse the $-$ following 6 to get

$$I(c) = 213 + 6 + 54.$$

We argue as in Theorem 2.3 that I is a sign-reversing involution, so that

$$\sum_{r=0}^s \sum_{c \in \overline{C}_{n, s, r}^{X, Y}} w(c) = \sum_{r=0}^s \sum_{\substack{c \in \overline{C}_{n, s, r}^{X, Y} \\ I(c) = c}} w(c).$$

Now, consider a fixed point c of I . As in the proof of Theorem 2.3, c can have no $-$'s, and thus $r = |X_n| - s$ and $w(c) = 1$. No string of multiple $+$'s can occur, since the first $+$ in such a string could be reversed. Thus, each of the $(|X_n| - s)$ $+$'s appears singly, and must either

- immediately follow some $c_i \in X$, $1 \leq i < n$, such that (c_i, c_{i+1}) is not an X, Y -descent pair of the underlying permutation, or
- immediately follow $c_n \in X$.

Thus $|X_n| - s$ elements of X_n immediately precede a $+$ that cannot be reversed, and are thus not the tops of X, Y -descent pairs. It follows that each of the remaining s elements of X_n do not immediately precede a $+$, and as such each must be the top of an X, Y -descent pair. Thus the underlying permutation $c_1 c_2 \cdots c_n$ has exactly s X, Y -descents.

Again, we observe that if $\sigma_1 \sigma_2 \cdots \sigma_n$ is a permutation with exactly s X, Y -descents, then we can create a fixed point of I by inserting a $+$ after every element of X_n that is not the top of an X, Y -descent pair. For example, if $X = \{2, 3, 4, 6, 8, 9\}$, $Y = \{1, 2, 3, 5\}$, $n = 9$, $s = 4$, and $\sigma = 958621437$, then the corresponding configuration would be

$$958 + 62143 + 7.$$

□

We note that the quantity $r + \beta_{X,n,x_i} - \beta_{Y,n,x_i}$ may be zero, or even negative. For example, let $X = \{2, 3, 4\}$, $Y = \{1, 3, 5\}$, $n = 6$, and $s = 3$. In constructing a $(6, 3, 0)^{X,Y}$ configuration, we start with an ordering of $X_6^c = \{1, 5, 6\}$, such as

$$516.$$

Since $|X_6| - s = 3 - 3 = 0$, there are no $+$'s to place, and the next step is to place the elements of X_6 in increasing order. There is one place to put $x_1 = 2$, namely, immediately before the 1. This corresponds to the fact that $\beta_{X,6,2} - \beta_{Y,6,2} = (2 - 1) - 0 = 1$. We then have the array

$$5216.$$

Notice that now there is no place to put $x_2 = 3$. There are no $+$'s, and 3 cannot be placed in front of 1 (the only element of Y smaller than 3) without violating condition (ii). This corresponds to the fact that $\beta_{X,6,3} - \beta_{Y,6,3} = (3 - 2) - 1 = 0$. So $P_{6,3}^{X,Y} = 0$, which we can also see by inspection: the only potential X, Y -descents are 21, 31, 41, and 43, and no permutation can contain more than one of 21, 31, and 41. Note, finally, that the quantity $r + \beta_{X,n,x_i} - \beta_{Y,n,x_i}$ is non-negative for $i = 1$, since $\beta_{Y,n,x_1} \leq \beta_{X,n,x_1} = x_1 - 1$, and that the difference

$$\begin{aligned} & (r + \beta_{X,n,x_i} - \beta_{Y,n,x_i}) - (r + \beta_{X,n,x_{i+1}} - \beta_{Y,n,x_{i+1}}) \\ &= (r + x_i - i - \beta_{Y,n,x_i}) - (r + x_{i+1} - (i + 1) - \beta_{Y,n,x_{i+1}}) \\ &= 1 + x_i - \beta_{Y,n,x_i} - (x_{i+1} - \beta_{Y,n,x_{i+1}}) \end{aligned}$$

is at most 1, since the sequence $\{x_i - \beta_{Y,n,x_i}\}_{i=1}^{|X_n|}$ is nondecreasing. Thus in a situation in which $r + \beta_{X,n,x_i} - \beta_{Y,n,x_i}$ is negative, we must have $r + \beta_{X,n,x_j} - \beta_{Y,n,x_j} = 0$ for some $j < i$.

Taking $Y = \mathbb{N}$ in (2.3) and (2.4) gives the following two corollaries.

Corollary 2.6.

$$P_{n,s}^X = |X_n^c|! \sum_{r=0}^s (-1)^{s-r} \binom{|X_n^c| + r}{r} \binom{n+1}{s-r} \prod_{x \in X_n} (1 + r + \alpha_{X,n,x}) \quad (2.5)$$

Corollary 2.7.

$$P_{n,s}^X = |X_n^c|! \sum_{r=0}^{|X_n^c| - s} (-1)^{|X_n^c| - s - r} \binom{|X_n^c| + r}{r} \binom{n+1}{|X_n^c| - s - r} \prod_{x \in X} (r + \beta_{X,n,x}) \quad (2.6)$$

Remark 2.8. Using a rook-placement interpretation outlined in Section 4 of this paper, (2.5) and (2.6) can also be obtained from formulas of Haglund [9] for hit numbers of Ferrers boards.

One can use (2.3) and (2.4) to obtain similar formulas for the coefficients of the polynomials $Q_n^X(x) = P_n^{\mathbb{N}, X}(x)$. Formulas for the coefficients of $Q_n^X(x)$ can also be derived directly from (2.5) and (2.6) by using the following result.

Theorem 2.9. *Given a subset $X \subseteq \mathbb{N}$ and a permutation $\sigma \in S_n$, let X^* be the subset of $[n]$ satisfying $i \in X^* \iff n+1-i \in X_n$. Then*

$$Q_{n,s}^X = P_{n,s}^{X^*}.$$

Proof. Given a permutation $\sigma = \sigma_1\sigma_2 \dots \sigma_n$, the *complement* of σ is $\sigma^c = (n+1-\sigma_1)(n+1-\sigma_2) \dots (n+1-\sigma_n)$. The *reverse* of σ is $\sigma^r = \sigma_n \dots \sigma_2\sigma_1$. The operations of “complement” and “reverse” are both clearly invertible. Now suppose σ has a descent pair (i, j) . Then σ^c has an ascent pair $(n+1-i, n+1-j)$, and so $(\sigma^c)^r$ has a descent pair $(n+1-j, n+1-i)$. Thus $\overrightarrow{des}_X(\sigma) = \overleftarrow{des}_{X^*}((\sigma^c)^r)$, which implies $Q_{n,s}^X = P_{n,s}^{X^*}$. \square

For example, if $X = k\mathbb{N}$ and $n = km + j$ for some $j, 0 \leq j \leq k-1$, then $X_n = \{k, 2k, \dots, km\} \subseteq [n]$ and $X^* = \{1+j, 1+j+k, \dots, 1+j+k(m-1)\}$. Thus $|(X^*)_n^c| = (k-1)m+j$, and, for $i = 0, \dots, m-1$, we have

$$\begin{aligned} \alpha_{X^*, n, 1+j+ik} &= km + j - (1+j+ik) - (m-1-i) = (k-1)(m-i), \text{ and} \\ \beta_{X^*, n, 1+j+ik} &= j + (k-1)i. \end{aligned}$$

It follows from Corollaries 2.6 and 2.7 that

$$\begin{aligned} Q_{km+j,s}^{k\mathbb{N}} &= P_{km+j,s}^{\{1+j, 1+j+k, \dots, 1+j+k(m-1)\}} = \\ &((k-1)m+j)! \sum_{r=0}^s (-1)^{s-r} \binom{(k-1)m+j+r}{r} \binom{km+j+1}{s-r} \prod_{i=1}^m (1+r+(k-1)i), \text{ and} \\ Q_{km+j,s}^{k\mathbb{N}} &= \\ &((k-1)m+j)! \sum_{r=0}^{m-s} (-1)^{m-s-r} \binom{(k-1)m+j+r}{r} \binom{km+j+1}{m-s-r} \prod_{i=0}^{m-1} (r+j+(k-1)i). \end{aligned}$$

These two formulas for $Q_{km+j,s}^{k\mathbb{N}}$ were first proved by Kitaev and Remmel in [16] using the special case of the recursion (2.1) with $X = \mathbb{N}$ and $Y = k\mathbb{N}$.

3. APPLICATIONS

Corollary 3.1. *Let $X = \mathbb{N}$, so that $\overleftarrow{Des}_X(\sigma) = Des(\sigma)$. Then*

$$P_{n,s}^X = \sum_{r=0}^s (-1)^{s-r} \binom{n+1}{s-r} (1+r)^n,$$

which is a well-known formula for the Eulerian numbers (see, e.g., [1], pp. 240–246).

In the following results we employ the notation of hypergeometric series. For $a \in \mathbb{R}$ and $n \in \mathbb{N}$, let $(a)_n = a(a+1)(a+2) \dots (a+n-1)$. Let $(a)_0 = 1$. Define

$${}_{m+1}F_m \left[\begin{matrix} a_0, & a_1, & a_2, & \dots, & a_m \\ & b_1, & b_2, & \dots, & b_m \end{matrix} \right] := \sum_{r=0}^{\infty} \frac{(a_0)_r (a_1)_r (a_2)_r \dots (a_m)_r}{r! (b_1)_r (b_2)_r \dots (b_m)_r}.$$

Since $(-n)_r = 0$ for all $r > n$, a hypergeometric series may be undefined if a parameter in the denominator is a negative integer. In our applications, *all* of the parameters are negative integers. However, in each case the largest (least negative) parameter occurs in the numerator, hence the series terminates in a well-defined, finite sum.

Corollary 3.2. *Let $X = 2\mathbb{N}$. Then*

$$P_{2n,s}^X = (n!)^2 \binom{n}{s},$$

which was originally derived by Kitaev and Remmel [15] using the special case of the recursion (2.1) with $X = 2\mathbb{N}$ and $Y = \mathbb{N}$.

Proof. By Corollary 2.6, we have

$$\begin{aligned}
P_{2n,s}^X &= n! \sum_{r=0}^s (-1)^{s-r} \binom{n+r}{r} \binom{2n+1}{s-r} \prod_{i=1}^n (i+r) \\
&= (s+1)_n^2 {}_3F_2 \left[\begin{matrix} -s, & -s, & -(2n+1) \\ & -(n+s), & -(n+s) \end{matrix} \right] \\
&= (s+1)_n^2 \frac{(n+1-s)_s (n+1-s)_s}{(n+1)_s (n+1)_s} \\
&= (n!)^2 \binom{n}{s}^2,
\end{aligned}$$

where in the third step we use the Pfaff-Saalschütz ${}_3F_2$ summation formula (see [7])

$${}_3F_2 \left[\begin{matrix} -n, & a, & b \\ & c, & a+b-c-n+1 \end{matrix} \right] = \frac{(c-a)_n (c-b)_n}{(c)_n (c-a-b)_n}.$$

□

Remark 3.3. More generally, if $X = \{u+2, u+4, u+6, \dots, u+2m\}$, a similar computation gives

$$P_{2m+u+v,s}^X = \binom{m}{s} \binom{m+u+v}{v+s} (m+u)! (m+v)!. \quad (3.1)$$

Combining (2.5) and (2.6) for various sets X , we get interesting identities, such as the following result of Kitaev and Remmel (Theorem 2 of [16]).

Corollary 3.4. Let $X = k\mathbb{N}$. Then for each $j, 0 \leq j \leq k-1$, we have

$$\begin{aligned}
P_{kn+j,s}^X &= \\
&((k-1)n+j)! \sum_{r=0}^s (-1)^{s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r} \prod_{i=0}^{n-1} (1+r+j+(k-1)i), \text{ and} \\
P_{kn+j,s}^X &= \\
&((k-1)n+j)! \sum_{r=0}^{n-s} (-1)^{n-s-r} \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{n-s-r} \prod_{i=1}^n (r+(k-1)i).
\end{aligned}$$

For some sets X the right-hand sides of (2.5) and (2.6) can be rewritten in terms of hypergeometric series; hence we obtain combinatorial proofs of identities such as the following.

Corollary 3.5. Let k and m be positive integers, and let s be a non-negative integer. Then

$$\begin{aligned}
&(s+1)_m^{k+1} {}_{k+2}F_{k+1} \left[\begin{matrix} -((k+1)m+1), & -s, & \dots, & -s \\ & -(m+s), & \dots, & -(m+s) \end{matrix} \right] = \\
&(km+1-s)_m^{k+1} {}_{k+2}F_{k+1} \left[\begin{matrix} -((k+1)m+1), & -(km-s), & \dots, & -(km-s) \\ & -((k+1)m-s), & \dots, & -((k+1)m-s) \end{matrix} \right].
\end{aligned}$$

Proof. Let $X = \{i : i \not\equiv 1 \pmod{k+1}\}$, and use (2.5) and (2.6) to compute $P_{(k+1)m,s}^X$. □

This identity is a special case of an integral form of a transformation of Karlsson-Minton type hypergeometric series due to Gasper [6]:

$$\begin{aligned}
&{}_{k+2}F_{k+1} \left[\begin{matrix} w, & x, & b_1+d_1, & \dots, & b_k+d_k \\ & x+c+1, & b_1, & \dots, & b_k \end{matrix} \right] = \\
&\frac{\Gamma(1+x+c)\Gamma(1-w)}{\Gamma(1+x-w)\Gamma(c+1)} \prod_{i=1}^k \frac{(b_i-x)_{d_i}}{(b_i)_{d_i}} \\
&\times {}_{k+2}F_{k+1} \left[\begin{matrix} -c, & x, & 1+x-b_1, & \dots, & 1+x-b_k \\ & x+1-w, & 1+x-b_1-d_1, & \dots, & 1+x-b_k-d_k \end{matrix} \right]
\end{aligned}$$

The following identity, while still a special case of Gasper's transformation, is more general.

Corollary 3.6. *Let $u = (u_1, \dots, u_k)$ be a weakly increasing array of non-negative integers, and $v = (v_1, \dots, v_k)$ an array of positive integers. Then for $n \geq \sum_{i=1}^k v_i + \max\{u_i + v_i : 1 \leq i \leq k\} - u_1$, we have*

$$\begin{aligned} & (s+1)_a (s+u_1+1)_{v_1} \cdots (s+u_k+1)_{v_k} \\ & \quad \times {}_{k+2}F_{k+1} \left[\begin{matrix} -(n+1), & -s, & -(s+u_1) & \dots, & -(s+u_k) \\ & -(s+a) & -(s+u_1+v_1) & \dots, & -(s+u_k+v_k) \end{matrix} \right] = \\ & (-1)^n (-n+s)_a (-n+s+u_1)_{v_1} \cdots (-n+s+u_k)_{v_k} \\ & \quad \times {}_{k+2}F_{k+1} \left[\begin{matrix} -(n+1), & -n+s+a & -n+s+u_1+v_1 & \dots, & -n+s+u_k+v_k \\ & -n+s & -n+s+u_1 & \dots, & -n+s+u_k \end{matrix} \right], \end{aligned} \tag{3.2}$$

where $a = n - \sum_{i=1}^k v_i$.

Remark 3.7. *Note that both hypergeometric series in (3.2) are balanced, i.e., the sum of the parameters in the top row is one less than the sum of the parameters in the bottom row. Balanced hypergeometric series are a particularly well-behaved class of hypergeometric series for which several summation and transformation results exist.*

Proof. For each $i, 1 \leq i \leq k$, let

$$f(i) = |\{j : u_j \leq i \leq u_j + v_j - 1\}|.$$

Define $M = \max\{m : f(m) > 0\}$ and set $b = a + 1 - M - u_1$. Let X be the subset of \mathbb{N} defined by the binary sequence

$$\tau = \underbrace{0 \dots 0}_{b} \underbrace{1 \dots 1}_{f(M)} 0 \underbrace{1 \dots 1}_{f(M-1)} 0 \underbrace{1 \dots 1}_{f(M-2)} 0 \dots 0 \underbrace{1 \dots 1}_{f(u_1+1)} 0 \underbrace{1 \dots 1}_{f(u_1)} \underbrace{0 \dots 0}_{u_1}.$$

That is, let $i \in X$ if and only if $\tau_i = 1$.

We prove the identity by showing that both sides are equal to $P_{n,s}^X$. Applying (2.5) gives

$$\begin{aligned} P_{n,s}^X &= a! \sum_{r=0}^s (-1)^{s-r} \binom{a+r}{r} \binom{n+1}{s-r} \prod_{i=u_1}^M (i+1+r)^{f(i)} \\ &= a! \sum_{r=0}^s (-1)^{s-r} \binom{a+r}{r} \binom{n+1}{s-r} \prod_{i=1}^k (u_i+1+r)_{v_i} \\ &= (s+1)_a (s+u_1+1)_{v_1} \cdots (s+u_k+1)_{v_k} \\ & \quad \times {}_{k+2}F_{k+1} \left[\begin{matrix} -(n+1), & -s, & -(s+u_1) & \dots, & -(s+u_k) \\ & -(s+a) & -(s+u_1+v_1) & \dots, & -(s+u_k+v_k) \end{matrix} \right]. \end{aligned}$$

The key observation here is that $\prod_{i=u_1}^M (i+1+r)^{f(i)} = \prod_{i=1}^k (u_i+1+r)_{v_i}$, since $i+1+r$ occurs in $\prod_{j=1}^k (u_j+1+r)_{v_j}$ exactly $f(i) = |\{j : u_j \leq i \leq u_j + v_j - 1\}|$ times.

On the other hand, applying (2.6) gives

$$\begin{aligned}
 P_{n,s}^X &= a! \sum_{r=0}^{n-a-s} (-1)^{n-a-s-r} \binom{a+r}{r} \binom{n+1}{n-a-s-r} \prod_{i=u_1}^M (a-i+r)^{f(i)} \\
 &= a! \sum_{r=0}^{n-a-s} (-1)^{n-a-s-r} \binom{a+r}{r} \binom{n+1}{n-a-s-r} \prod_{i=1}^k (a+1-u_i-v_i+r)^{v_i} \\
 &= (-1)^n (-n+s)_a (-n+s+u_1)_{v_1} \cdots (-n+s+u_k)_{v_k} \\
 &\quad \times {}_{k+2}F_{k+1} \left[\begin{matrix} -(n+1), & -n+s+a & -n+s+u_1+v_1 & \dots, & -n+s+u_k+v_k \\ & -n+s & -n+s+u_1 & \dots, & -n+s+u_k \end{matrix} \right].
 \end{aligned}$$

□

Example 3.8. Let $u = (0, 1, 1, 5), v = (2, 3, 1, 2)$, and $n = 16$. If we represent u and v with four rows of \bullet 's, the i^{th} row starting at u_i and having length v_i , then $f(i) = |\{j : u_j \leq i \leq u_j + v_j - 1\}|$ is the number of \bullet 's in the i^{th} column. In this example we have $f(0) = 1, f(1) = 3, f(2) = f(3) = 1, f(4) = 0$, and $f(5) = f(6) = 1$, as shown below.

$$\begin{array}{cccccccc}
 & & & & & \bullet & \bullet & \\
 & & & & & & & \\
 & & & \bullet & & & & \\
 & & \bullet & \bullet & \bullet & & & \\
 \bullet & \bullet & & & & & & \\
 0 & 1 & 2 & 3 & 4 & 5 & 6 &
 \end{array}$$

The corresponding binary sequence is

$$\tau = 0010100101011101,$$

and the corresponding set is $X = \{3, 5, 8, 10, 12, 13, 14, 16\}$. Using (2.5) and (2.6) to compute $P_{n,s}^X$ gives

$$\begin{aligned}
 &(s+1)_8 (s+1)_2 (s+2)_3 (s+2)_1 (s+6)_2 \\
 &\quad \times {}_6F_5 \left[\begin{matrix} -17, & -s, & -s, & -(s+1), & -(s+1), & -(s+5) \\ & -(s+8), & -(s+2), & -(s+4), & -(s+2), & -(s+7) \end{matrix} \right] \\
 &= (-1)^{16} (-16+s)_8 (-16+s)_2 (-15+s)_3 (-15+s)_1 (-11+s)_2 \\
 &\quad \times {}_6F_5 \left[\begin{matrix} -17, & -8+s, & -14+s, & -12+s, & -14+s, & -9+s \\ & -16+s, & -16+s, & -15+s, & -15+s, & -11+s \end{matrix} \right].
 \end{aligned}$$

4. CONNECTIONS WITH ROOK THEORY

A *board* is a finite subset of an infinite grid of unit squares. The “rook number” $r_k(B)$ of a board B is defined to be the number of ways to place k non-attacking rooks on B . Two boards B and B' are *rook-equivalent* if $r_k(B) = r_k(B')$ for all k . For a board B contained inside the $n \times n$ board, the “hit number” $h_k(B)$ is defined to be the number of ways to place n non-attacking rooks on the $n \times n$ board so that exactly k rooks lie on B . In what follows we will focus on hit numbers rather than rook numbers. Kaplansky and Riordan [11] showed that rook-equivalent boards have the same hit numbers.

The key to the connection between rook placements and descents of permutations is Foata’s First Transformation [4], a bijection $\Phi : S_n \rightarrow S_n$ which exchanges excedences and descents. An *excedence* of $\sigma = \sigma_1 \sigma_2 \dots \sigma_n$ is an entry σ_i satisfying $\sigma_i > i$. Foata’s transformation can most easily be explained with an example.

Example 4.1. Let $\omega = 61437258 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 6 & 1 & 4 & 3 & 7 & 2 & 5 & 8 \end{pmatrix}$. This permutation has three excedences: $\frac{1}{6}$, $\frac{3}{4}$, and $\frac{5}{7}$. The first step in Foata’s transformation is to write ω in cycle form: $(162)(34)(57)(8)$. Next, write each cycle with largest element last, and order the cycles by increasing

largest element: $(34)(216)(57)(8)$. Finally, to compute $\Phi(\omega)$, reverse each cycle and erase the parentheses: $\Phi(\omega) = 43612758$. In this example the descents of $\Phi(\omega)$ are 43, 61, and 75. In general, it is not hard to see that (i, j) is a descent pair of $\Phi(\omega)$ if and only if $\binom{j}{i}$ is an excedence of ω . To go backwards, given $\sigma = 43612758$, cut before each left-to-right maxima: $43|612|75|8$, then reverse each block to get the cycles of $\Phi^{-1}(\sigma)$: $(34)(216)(57)(8)$.

Foata's transformation is key to this section because rook placements provide a convenient way of tracking the excedences of a permutation. As the following example illustrates, given any subset $U \subseteq \{(i, j) : 1 \leq j < i \leq n\}$ of potential excedences we can construct a board B_n^U inside the $n \times n$ board so that the number of $\sigma \in S_n$ with exactly s U -excedences, and hence the number of $\sigma \in S_n$ with exactly s U -descents, is $h_s(B_n^U)$.

Example 4.2. Suppose we wish to count descents $\sigma_i > \sigma_{i+1}$ satisfying $\sigma_i \in E, \sigma_{i+1} \in O$, and $\sigma_i - \sigma_{i+1} \in \{1, 3\}$ (this is an instance of counting what we have called “ X, Y, Z -descents”). For $n = 8$, the board B_8^U consists of the squares $(i, j) \in [8] \times [8]$ such that $i \in E, j \in O$, and $i - j = 1$ or 3. We have pictured this board as the shaded squares in Figure 1. Now consider the placement,

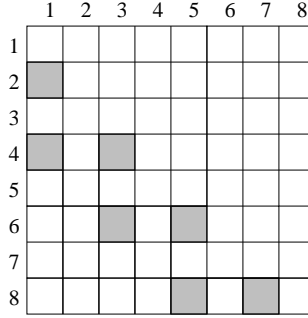


FIGURE 1. The board B_8^U .

shown in Figure 2, of eight non-attacking rooks (marked by X 's) on the 8×8 board so that two rooks lie on B_8^U . This placement corresponds to the permutation $\omega = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 1 & 5 & 7 & 6 & 2 & 3 & 8 \end{pmatrix}$,

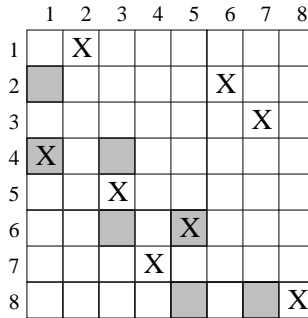


FIGURE 2. A placement of rooks on the 8×8 board.

with the rooks placed on B_8^U corresponding to the excedences $\frac{1}{4}$ and $\frac{5}{6}$. We now employ Foata's transformation to get the permutation $\sigma = \Phi(\omega) = 74126538$ with exactly two U -descents: 41 and 65.

One important class of boards is the class of *Ferrers* boards, that is, boards of partition shape. Ferrers boards are usually drawn right justified, as in Figure 3. For $X, Y \subseteq [n]$, the board $B_n^{X,Y}$ corresponding to the potential X, Y -descents of permutations $\sigma \in S_n$ is trivially rook-equivalent to

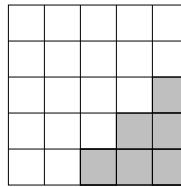


FIGURE 3. A Ferrers board.

a Ferrers board; we need only shift all the non-empty rows and columns to the bottom-left of the square and take the mirror image. For example, if $X = \{2, 3, 5\}, Y = \{1, 2, 4\}$, and $n = 5$, the board $B_5^{X,Y}$ is shown in Figure 4; this board is trivially rook-equivalent to the Ferrers board shown in Figure 3. Therefore, in certain cases we can make use of results for Ferrers boards in computing the

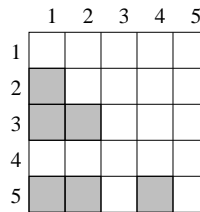


FIGURE 4. The board $B_5^{X,Y}$.

numbers $P_{n,s}^{X,Y}$. As one example, in [9], Haglund gives several formulas involving the hit numbers of Ferrers boards, one of which can be specialized to obtain (2.5).

As another example, we can use the rook interpretation of our problem to give a purely combinatorial proof of Corollary 3.2. For instance, the board $B_8^{E,\mathbb{N}}$ corresponding to even descents of permutations $\sigma \in S_8$ is shown in Figure 5. The corresponding Ferrers board is shown in Figure 6.

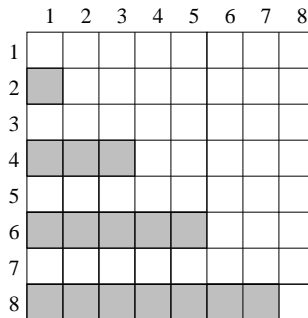
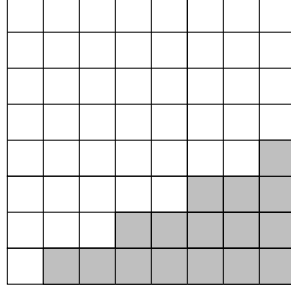


FIGURE 5. The board $B_8^{E,\mathbb{N}}$.

In what follows we use the notation of [5] and [8]. Let the column heights of a Ferrers board B inside an $n \times n$ square be given by the “height vector” $h(B) = (h_1, h_2, \dots, h_n)$. Define the “structure vector” $s(B) = (s_1, s_2, \dots, s_n)$, where $s_i = h_i - (i - 1), 1 \leq i \leq n$. Here it is standard practice to insist that n be large enough so that none of the entries of the structure vector is positive. This can always be done, for example, by taking n greater than the number of squares of B . In our applications n is already fixed; however, the entries of the structure vector are still non-positive because boards corresponding to descents necessarily lie strictly below the main diagonal. In [5], Foata and Schützengerger showed that two Ferrers boards B and B' are rook-equivalent if and only if the entries of $s(B)$ and $s(B')$ are equal as multisets.

In our example, we first compute $s(B)$:

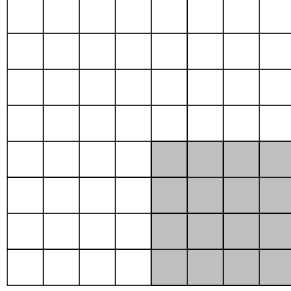
FIGURE 6. The Ferrers board B corresponding to $B_8^{E,\mathbb{N}}$.

$$\begin{array}{cccccccc} 0 & 1 & 1 & 2 & 2 & 3 & 3 & 4 \\ - & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 0 & -1 & -1 & -2 & -2 & -3 & -3 \end{array}$$

Thus B is rook-equivalent to the board B' with structure vector $s(B') = (0, -1, -2, -3, 0, -1, -2, -3)$. To identify B' we next compute $h(B')$:

$$\begin{array}{cccccccc} 0 & -1 & -2 & -3 & 0 & -1 & -2 & -3 \\ + & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 0 & 0 & 0 & 0 & 4 & 4 & 4 & 4 \end{array}$$

Thus B' is the board shown in Figure 7.

FIGURE 7. The board B' , rook-equivalent to B .

In general, the board for even descents of permutations $\sigma \in S_{2n}$ has structure vector

$$s(B) = (0, 0, -1, -1, \dots, -(n-1), -(n-1)),$$

and is thus rook-equivalent to the square $n \times n$ board B' , which has structure vector

$$s(B') = (0, -1, \dots, -(n-1), 0, -1, \dots, -(n-1)).$$

The hit number h_s for this square board B' inside the $2n \times 2n$ board is given by

$$h_s = \binom{n}{s}^2 s! \cdot \binom{n}{n-s} (n-s)! \cdot n! = (n!)^2 \binom{n}{s}^2,$$

since we can first place s rooks on the $n \times n$ board B' in $\binom{n}{s}^2 s!$ ways, then place $n-s$ rooks above B' in $\binom{n}{n-s} (n-s)!$ ways, and finally place n rooks in the left half of the $2n \times 2n$ board in $n!$ ways.

An important class of boards whose hit numbers have a simple product formula is the class of rectangular boards. One might ask whether there are other sets X such that the board corresponding to X -descents is rook equivalent to a rectangular board. In fact, Remark 3.3 (in which $X = \{u+2, u+4, u+6, \dots, u+2m\}$) covers all possibilities. For example, consider the 2×3 rectangular board B shown in Figure 8. We place this board in the lower right corner of an 8×8 board, as shown in Figure 9.

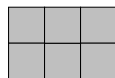


FIGURE 8. The 2×3 board B .

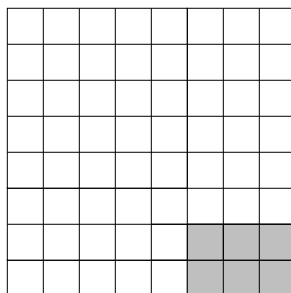


FIGURE 9. The board B placed in an 8×8 board.

Note that for any set X the board associated to X -descents has distinct rows, since the row corresponding to $i \in X$ has length $i - 1$. Rearranging the elements of $s(B)$ in weakly decreasing order gives the unique (see [5]) board B' with distinct rows that is rook-equivalent to B . In our example, we compute $s(B) = (0, -1, -2, -3, -4, -3, -4, -5)$. Thus B is rook-equivalent to the board B' with structure vector $s(B') = (0, -1, -2, -3, -3, -4, -4, -5)$ and height vector $h(B') = (0, 0, 0, 0, 1, 1, 2, 2)$. B' is the board shown in Figure 10. Finally, to get the board for X -descents,

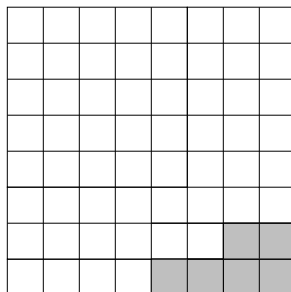


FIGURE 10. The board B' , rook-equivalent to B .

we take the mirror image, and shift the rows upwards so that a row of length $i - 1$ is in position i , as shown in Figure 11. Thus in our example, $X = \{3, 5\}$. In general, if we start with a rectangular

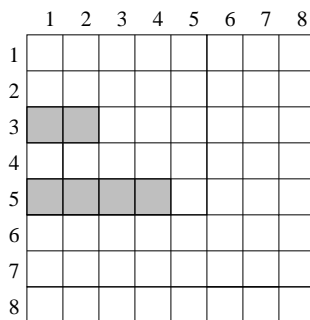


FIGURE 11. The board B_8^X .

$a \times b$ board ($a \leq b$), then the corresponding set is $X = \{u + 2, u + 4, u + 6, \dots, u + 2m\}$, where $m = a$

and $u = b - a$. For $n = 2m + u + v$, we get

$$\begin{aligned} P_{n,s}^X &= \binom{a}{s} \binom{b}{s} s! \cdot \binom{n-a}{b-s} (b-s)! \cdot (n-b)! \\ &= \binom{m}{s} \binom{m+u}{s} s! \cdot \binom{m+u+v}{m+u-s} (m+u-s)! \cdot (m+v)! \\ &= \binom{m}{s} \binom{m+u+v}{v+s} (m+u)! (m+v)!, \end{aligned}$$

exactly as in (3.1).

As noted in the introduction, we can also use the rook interpretation to translate the more general problem of counting X, Y -descents into one for which Corollaries 2.6 and 2.7 apply directly.

Proposition 4.3. *Given subsets $X, Y \subseteq \mathbb{N}$ and a permutation $\sigma \in S_n$, let B be the Ferrers board corresponding to the potential descent pairs (i, j) , where $i \in X_n$ and $j \in Y_n$. Let B' be the unique Ferrers board rook-equivalent to B that has distinct rows. Let $X' \subseteq [n]$ be the unique subset whose corresponding board (once empty rows and columns are deleted, and taking the mirror image) is B' . Then*

$$P_{n,s}^{X,Y} = P_{n,s}^{X'}.$$

Proof. By the previous discussion, we have $P_{n,s}^{X,Y} = h_s(B)$ and $P_{n,s}^{X'} = h_s(B')$. But B and B' are rook-equivalent, and thus $h_s(B) = h_s(B')$ for all s . \square

Example 4.4. *Let $X = \{2, 3, 5, 7, 8\}$, $Y = \{1, 2, 4, 5, 6\}$, and $n = 8$, so that the potential descent pairs are 21, 31, 32, 51, 52, 54, 71, 72, 74, 75, 76, 81, 82, 84, 85, and 86. Then $P_{8,s}^{X,Y} = h_s(B_8^{X,Y})$, where $B_8^{X,Y}$ is the board shown in Figure 12. The unique board B' rook-equivalent to $B_8^{X,Y}$ that has distinct*

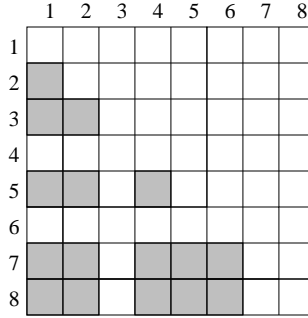


FIGURE 12. The board $B_8^{X,Y}$.

rows is shown in Figure 13. B' is the board for $X' = \{2, 3, 4, 5, 7\}$, and so $P_{8,s}^{X,Y} = P_{8,s}^{X'}$ for all s .

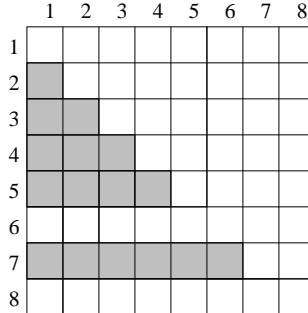


FIGURE 13. The board B' , rook-equivalent to $B_8^{X,Y}$.

5. WORDS

Our results (2.3) and (2.4) extend easily to words. Let $\rho = (\rho_1, \rho_2, \dots, \rho_m)$ be a composition of n , and let $R(\rho)$ be the rearrangement class of the word $1^{\rho_1}2^{\rho_2}\dots m^{\rho_m}$ (i.e., ρ_1 copies of 1, ρ_2 copies of 2, etc.). Given $X, Y \subseteq \mathbb{N}$, and a word $w \in R(\rho)$, define

$$\begin{aligned} Des_{X,Y}(w) &= \{i : w_i > w_{i+1} \text{ \& } w_i \in X \text{ \& } w_{i+1} \in Y\}, \\ des_{X,Y}(w) &= |Des_{X,Y}(w)|, \text{ and} \\ P_{\rho,s}^{X,Y} &= |\{w \in R(\rho) : des_{X,Y}(w) = s\}|. \end{aligned}$$

Then

Theorem 5.1.

$$P_{\rho,s}^{X,Y} = \binom{a}{\rho_{v_1}, \rho_{v_2}, \dots, \rho_{v_b}} \sum_{r=0}^s (-1)^{s-r} \binom{a+r}{r} \binom{n+1}{s-r} \prod_{x \in X} \binom{\rho_x + r + \alpha_{X,\rho,x} + \beta_{Y,\rho,x}}{\rho_x}, \quad (5.1)$$

where $X_m^c = \{v_1, v_2, \dots, v_b\}$, $a = \sum_{i=1}^b \rho_{v_i}$, and for any $x \in X_m$,

$$\begin{aligned} \alpha_{X,\rho,x} &= \sum_{\substack{z \notin X \\ x < z \leq m}} \rho_z, \text{ and} \\ \beta_{Y,\rho,x} &= \sum_{\substack{z \notin Y \\ 1 \leq z < x}} \rho_z. \end{aligned}$$

Proof. We proceed as in the proof of Theorem 2.3. Fix s and the composition $\rho = (\rho_1, \rho_2, \dots, \rho_m)$ of n . Given r such that $0 \leq r \leq s$, a $(\rho, s, r)^{X,Y}$ -configuration c consists of an array of the elements of the multiset $\{1^{\rho_1}, 2^{\rho_2}, \dots, m^{\rho_m}\}$, r $+$'s, and $(s-r)$ $-$'s, satisfying

- (i) each $-$ is either at the very beginning of the array or immediately follows a number, and
- (ii) if x and y are consecutive numbers in the array, with $x \in X, y \in Y$, and $x > y$, i.e., if (x, y) forms an X, Y -descent pair in the underlying word, then there must be at least one $+$ between x and y .

As an example, if $\rho = (2, 3, 1, 4, 2)$, $X = \{2, 3, 5\}$, and $Y = \{1, 3, 4\}$, the following is a $(\rho, 5, 3)^{X,Y}$ -configuration.

$$c = 4124 - 413 - +25 + 42 + 5$$

In this example, the underlying word is 412441325425. As before, we will let $c_1 c_2 \dots c_n$ denote the underlying word of the $(\rho, s, r)^{X,Y}$ -configuration c .

Let $C_{\rho,s,r}^{X,Y}$ be the set of all $(\rho, s, r)^{X,Y}$ -configurations. Then we claim that

$$|C_{\rho,s,r}^{X,Y}| = \binom{a}{\rho_{v_1}, \rho_{v_2}, \dots, \rho_{v_b}} \binom{a+r}{r} \binom{n+1}{s-r} \prod_{x \in X} \binom{\rho_x + r + \alpha_{X,\rho,x} + \beta_{Y,\rho,x}}{\rho_x}.$$

That is, we can construct the set of $(\rho, s, r)^{X,Y}$ -configurations as follows. First, we order the elements of the multiset $\{v_1^{\rho_{v_1}}, \dots, v_b^{\rho_{v_b}}\}$. This can be done in $\binom{a}{\rho_{v_1}, \rho_{v_2}, \dots, \rho_{v_b}}$ ways. Next, we insert the r $+$'s. This can be done in $\binom{a+r}{r}$ ways. Writing $X_m = \{x_1 < x_2 < \dots < x_{|X_m|}\}$, we can next place the elements of the multiset $\{x_1^{\rho_{x_1}}, \dots, x_{|X_m|}^{\rho_{x_{|X_m|}}}\}$ in $\prod_{i=1}^{|X_m|} \binom{\rho_{x_i} + r + \alpha_{X,\rho,x_i} + \beta_{Y,\rho,x_i}}{\rho_{x_i}}$ ways, since after placing all copies of x_1, x_2, \dots, x_{i-1} , the ρ_{x_i} copies of x_i can either go

- immediately before any of the β_{Y,ρ,x_i} elements of $\{1, 2, \dots, x_{i-1}\}$ that is not in Y , or
- immediately before any of the α_{X,ρ,x_i} elements of $\{x_i + 1, x_i + 2, \dots, m\}$ that is not in X , or
- immediately before any of the r $+$'s, or
- at the very end of the array.

Thus, we have $1 + r + \alpha_{X,\rho,x_i} + \beta_{Y,\rho,x_i}$ places in which we can insert letters equal to x_i . The number of ways in which we can place the x_i 's is therefore equal to the number of positive integral solutions of the equation

$$z_1 + \cdots + z_{1+r+\alpha_{X,\rho,x_i}+\beta_{Y,\rho,x_i}} = \rho_{x_i},$$

which is well known to be $\binom{\rho_{x_i} + r + \alpha_{X,\rho,x_i} + \beta_{Y,\rho,x_i}}{\rho_{x_i}}$. Finally, we can place the $-$'s in $\binom{n+1}{s-r}$ ways.

As before, we define the weight $w(c)$ of an $(\rho, s, r)^{X,Y}$ -configuration c to be -1 to the number of $-$'s of c . It then follows that the RHS of (5.1) equals

$$\sum_{r=0}^s \sum_{c \in C_{\rho,s,r}^{X,Y}} w(c)$$

We now employ the identical involution I as in the proof of Theorem 2.3, this time on the set $C_{\rho,s}^{X,Y} = \bigsqcup_{r=0}^s C_{\rho,s,r}^{X,Y}$. As before, we scan from left to right, and reverse the first sign that we can reverse without violating conditions (i) and (ii). In our example above, the first place where we either encounter a sign that can be reversed is the $-$ after the second 4. Thus

$$I(c) = 4124 + 413 - +25 + 42 + 5.$$

As was the case with the proof of Theorem 2.3, it is simple to check that I is a sign-reversing involution.

Now, suppose that $I(c) = c$. Then c clearly can have no $-$'s, and so $r = s$ and $w(c) = 1$. It must also be the case that no $+$'s can be reversed. Thus each of the s $+$'s must occur singly in the middle of an X, Y -descent pair. It follows that the underlying word has exactly s X, Y -descents. \square

Taking $Y = \mathbb{N}$ in (5.1) gives the word analogue of Corollary 2.6.

Corollary 5.2.

$$P_{\rho,s}^X = \binom{a}{\rho_{v_1}, \rho_{v_2}, \dots, \rho_{v_b}} \sum_{r=0}^s (-1)^{s-r} \binom{a+r}{r} \binom{n+1}{s-r} \prod_{x \in X} \binom{\rho_x + r + \alpha_{X,\rho,x}}{\rho_x}, \quad (5.2)$$

where we write $P_{\rho,s}^X$ for $P_{\rho,s}^{X,\mathbb{N}}$.

Corollary 5.3. *Let $X = \{2\}$ and $\rho = (a, b)$. Then*

$$P_{\rho,s}^X = \binom{a}{s} \binom{b}{s}.$$

Proof. By (5.2) we have

$$\begin{aligned} P_{\rho,s}^X &= \binom{a}{a} \sum_{r=0}^s (-1)^{s-r} \binom{a+r}{r} \binom{a+b+1}{s-r} \binom{b+r}{b} \\ &= \frac{(s+1)_a}{a!} \frac{(s+1)_b}{b!} {}_3F_2 \left[\begin{matrix} -s, & -s, & -(a+b+1) \\ & -(a+s), & -(b+s) \end{matrix} \right] \\ &= \frac{(s+1)_a}{a!} \frac{(s+1)_b}{b!} \frac{(a-s+1)_s (b-s+1)_s}{(a+1)_s (b+1)_s} \\ &= \binom{a}{s} \binom{b}{s}. \end{aligned}$$

\square

Remark 5.4. *We can give a purely combinatorial proof of Corollary 5.3 using Foata's transformation switching descents and excedences. The number of rearrangements of a 1's and b 2's with exactly s excedences is $\binom{a}{s} \binom{b}{s}$, since we have to choose which s of the first a spots to be 2's (giving s excedences) and which s of the last b spots to be 1's.*

Corollary 5.5. *Let $X = 2\mathbb{N}$ and $\rho = \underbrace{(k, k, \dots, k)}_{2n}$. Then*

$$P_{\rho, s}^X = \binom{kn}{k, k, \dots, k}^2 \binom{kn}{s}^2.$$

Proof. By (5.2) we have

$$\begin{aligned} P_{\rho, s}^X &= \binom{kn}{k, k, \dots, k} \sum_{r=0}^s (-1)^{s-r} \binom{kn+r}{r} \binom{2kn+1}{s-r} \prod_{i=1}^n \binom{ki+r}{k} \\ &= \frac{1}{(k!)^{2n}} \sum_{r=0}^s (-1)^{s-r} (r+1)_{kn}^2 \binom{2kn+1}{s-r} \\ &= \frac{(s+1)_{kn}^2}{(k!)^{2n}} {}_3F_2 \left[\begin{matrix} -s, & -s, & -(2kn+1) \\ & -(kn+s), & -(kn+s) \end{matrix} \right] \\ &= \binom{kn}{k, k, \dots, k}^2 \binom{kn}{s}^2. \end{aligned}$$

□

Finally, there is an alternative formula for $P_{\rho, s}^{X, Y}$. In addition to the notation of Theorem 5.1, let

$$\beta_{X, \rho, x} = \sum_{\substack{z \notin X \\ 1 \leq z < x}} \rho_z.$$

We have

Theorem 5.6.

$$P_{\rho, s}^{X, Y} = \binom{a}{\rho_{v_1}, \rho_{v_2}, \dots, \rho_{v_b}} \sum_{r=0}^{n-a-s} (-1)^{n-a-s-r} \binom{a+r}{r} \binom{n+1}{n-a-s-r} \prod_{x \in X} \binom{r + \beta_{X, \rho, x} - \beta_{Y, \rho, x}}{\rho_x}, \quad (5.3)$$

where we use the convention that $\binom{p}{q} = 0$ if $p < 0$.

Proof. The proof is analogous to that of Theorem 2.5. Given a fixed composition $\rho = (\rho_1, \rho_2, \dots, \rho_m)$ of n , and $s \geq r \geq 0$, a $\overline{(\rho, s, r)}^{X, Y}$ -configuration is an array of the elements of the multiset $\{1^{\rho_1}, 2^{\rho_2}, \dots, m^{\rho_m}\}$, together with r +’s and $(n - a - s - r)$ -’s, satisfying

- (i) each - is either at the very beginning of the string or immediately follows a number,
- (ii) if $c_i \in X$, $1 \leq i < n$, and (c_i, c_{i+1}) is not an X, Y -descent pair of the underlying word, then there must be at least one + between c_i and c_{i+1} , and
- (iii) if $c_n \in X$, then c_n must be followed by at least one +.

As an example, if $X = \{2, 3, 6\}$, $Y = \{1, 2, 5\}$, and $\rho = (2, 1, 3, 2, 1, 1)$, then the following is a $\overline{(\rho, 1, 2)}^{X, Y}$ -configuration.

$$213 + 43 + 3 - 165 - 4$$

Let $\overline{C}_{\rho, s, r}^{X, Y}$ be the set of all $\overline{(\rho, s, r)}^{X, Y}$ -configurations. Then we claim that

$$\left| \overline{C}_{\rho, s, r}^{X, Y} \right| = \binom{a+r}{r} \binom{a}{\rho_{v_1}, \rho_{v_2}, \dots, \rho_{v_b}} \binom{n+1}{n-a-s-r} \prod_{x \in X} \binom{r + \beta_{X, \rho, x} - \beta_{Y, \rho, x}}{\rho_x}.$$

That is, we can construct the set of $\overline{(\rho, s, r)}^{X, Y}$ -configurations as follows. First, we order the elements of the multiset $\{v_1^{\rho_{v_1}}, \dots, v_b^{\rho_{v_b}}\}$. This can be done in $\binom{a}{\rho_{v_1}, \rho_{v_2}, \dots, \rho_{v_b}}$ ways. Next, we insert the r +’s. This can be done in $\binom{a+r}{r}$ ways. Next, consider the choices for placing the elements of the multiset $\{x_1^{\rho_{x_1}}, x_2^{\rho_{x_2}}, \dots, x_{|X_m^c|}^{\rho_{x_{|X_m^c|}}}\}$ in the order $x_1 < x_2 < \dots < x_{|X_m^c|}$. First, there are $r + \beta_{X, \rho, x_1} - \beta_{Y, \rho, x_1}$ spaces in which to insert the x_1 ’s, since x_1 can either go immediately before of any +, or immediately before any element of Y which is less than x_1 . Note that unlike the situation in Theorem 5.1, no

more than one copy of x_1 can go in any particular available space. Thus there are $\binom{r+\beta_{X,\rho,x_1}-\beta_{Y,\rho,x_1}}{\rho_{x_1}}$ ways to place the x_1 's.

In general, having placed all copies of x_1, x_2, \dots, x_{i-1} , we cannot place x_i immediately before some $y \in Y, y < x_i$, which earlier had an element of the multiset $\{x_1^{\rho_{x_1}}, x_2^{\rho_{x_2}}, \dots, x_{i-1}^{\rho_{x_{i-1}}}\}$ placed immediately before it. Similarly, we cannot place x_i immediately before any $+$ which earlier had an element of the multiset $\{x_1^{\rho_{x_1}}, x_2^{\rho_{x_2}}, \dots, x_{i-1}^{\rho_{x_{i-1}}}\}$ placed immediately before it. It then follows that there are

$$r + \sum_{1 \leq z < x_i} \rho_z - \beta_{Y,\rho,x_i} - \sum_{j=1}^{i-1} \rho_{x_j} = r + \beta_{X,\rho,x_i} - \beta_{Y,\rho,x_i}$$

spaces in which to insert the x_i 's. Thus, there are total of $\prod_{i=1}^{|X_m^c|} \binom{r+\beta_{X,\rho,x_i}-\beta_{Y,\rho,x_i}}{\rho_{x_i}}$ ways to place all of the copies of $x_1, x_2, \dots, x_{|X_m^c|}$, given our placement of the copies of v_1, v_2, \dots, v_b . Finally, we can place the $-$'s in $\binom{n+1}{n-a-s-r}$ ways.

The remainder of the proof follows exactly as in Theorem 2.5. The weight of a configuration is defined in the same way, and applying the same sign-reversing involution I cancels out all configurations except those corresponding to words with exactly s X, Y -descent pairs. \square

In the special case $Y = \mathbb{N}$, Theorem 5.6 reduces to the following.

Corollary 5.7.

$$P_{\rho,s}^X = \binom{a}{\rho_{v_1}, \rho_{v_2}, \dots, \rho_{v_b}} \sum_{r=0}^{n-a-s} (-1)^{n-a-s-r} \binom{a+r}{r} \binom{n+1}{s-r} \prod_{x \in X} \binom{r+\beta_{X,\rho,x}}{\rho_x}.$$

6. X, Y, Z -DESCENTS

As mentioned in the Introduction, a more general problem is to study the class of polynomials

$$P_n^{X,Y,Z}(x) = \sum_{s \geq 0} P_{n,s}^{X,Y,Z} x^s := \sum_{\sigma \in S_n} x^{\text{des}_{X,Y,Z}(\sigma)},$$

where for any subsets X, Y , and Z of \mathbb{N} , and permutation $\sigma = \sigma_1 \sigma_2 \cdots \sigma_n \in S_n$,

$$\begin{aligned} \text{Des}_{X,Y,Z}(\sigma) &= \{i : \sigma_i > \sigma_{i+1} \text{ \& } \sigma_i \in X, \sigma_{i+1} \in Y \text{ \& } \sigma_i - \sigma_{i+1} \in Z\}, \text{ and} \\ \text{des}_{X,Y,Z}(\sigma) &= |\text{Des}_{X,Y,Z}(\sigma)|. \end{aligned}$$

We say shall that (σ_i, σ_{i+1}) is an X, Y, Z -descent if $i \in \text{Des}_{X,Y,Z}(\sigma)$. The polynomials studied in this paper are thus the special case $Z = \mathbb{N}$ of the polynomials $P_n^{X,Y,Z}(x)$.

In many cases, we can obtain formulas for the coefficients $P_{n,s}^{X,Y,Z}$ from our previous formulas. That is, in many cases, the possible X, Y, Z -descents under Foata's transformation corresponds to a board which is rook equivalent to a Ferrers board. In such cases, we can use the formulas for hit polynomials or our formulas for $P_{n,s}^{X,Y}$ or $P_{n,s}^X$ to obtain formulas for $P_{n,s}^{X,Y,Z}$. For example, let $E = 2\mathbb{N}$ and $\mathbb{N}_{\geq k} = \{k, k+1, k+2, \dots\}$. Under Foata's transformation, the $E, E, \mathbb{N}_{\geq 2k}$ -descents correspond to excedences of the form $\begin{smallmatrix} 2s \\ 2t \end{smallmatrix}$ where $2t - 2s \geq 2k$. For example, if $k = 2$ and $n = 20$, then we would consider the

board $B_{20}^{E,E,\mathbb{N}_{\geq 4}}$ shown in Figure 14. It is then easy to see that $B_{20}^{E,E,\mathbb{N}_{\geq 4}}$ is rook equivalent to the board $B_{20}^{\{2,3,4,5,6,7,8,9\}}$ pictured in Figure 15, which is the board we would consider when computing the polynomial $P_{20}^{\{2,3,4,5,6,7,8,9\}}(x)$. It follows that $P_{20}^{E,E,\mathbb{N}_{\geq 4}}(x) = P_{20}^{\{2,3,4,5,6,7,8,9\}}(x)$. Thus we can use Corollary 2.6 or Corollary 2.7 to give explicit formulas for $P_{20,s}^{E,E,\mathbb{N}_{\geq 4}} = P_{20,s}^{\{2,3,4,5,6,7,8,9\}}$.

The problem of computing $P_{n,s}^{X,Y,Z}$ for arbitrary sets X, Y , and Z seems to be difficult in large part because the board corresponding to X, Y, Z -descents is not a Ferrers board in general. This can be seen in Example 4.2, in which $X = E, Y = O$, and $Z = \{1, 3\}$. In some special cases, however, the rook-placement formulation will enable us to derive a formula for $P_{n,s}^{X,Y,Z}$ involving a double or triple sum. One such example is given below.

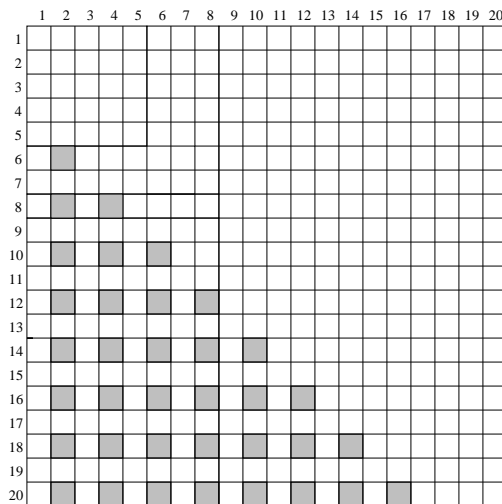


FIGURE 14. The board $B_{20}^{E,E,N \geq 4}$.

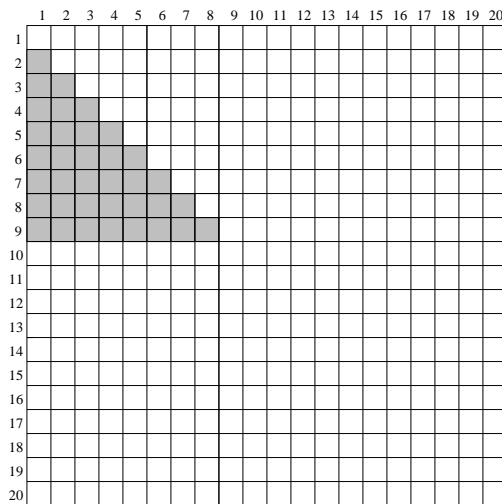


FIGURE 15. The board $B_{20}^{\{2,3,4,5,6,7,8,9\}}$.

Example 6.1. Let $X = \{z : z = 4, 5, \text{ or } 6 \pmod 6\}$, $Y = \{z : z = 1, 2, \text{ or } 3 \pmod 6\}$, and $Z = \{1, 2, 3, 4, 5, 6\}$. For $n = 12$, the board $B_{12}^{X,Y,Z}$ corresponding to X, Y, Z -descents is shown in Figure 16. By permuting rows and columns we see that $B_{12}^{X,Y,Z}$ is rook equivalent to the board B' shown in Figure 17. To compute the hit number h_s of B' , we think of placing the rooks in several successive steps, as indicated by the numbers on the diagram in Figure 18.

Suppose we first place p rooks on the lower-left 3×3 block of B' and q rooks on the upper-right 3×3 block of B' , where $p + q = s$. This first step can be done in $\binom{3}{p} \binom{3}{p} p! \cdot \binom{3}{q} \binom{3}{q} q!$ ways. Next, we place a total of $3 - q$ rooks in the regions marked '2'; this can be done in $\binom{9-p}{3-q} (3 - q)!$ ways. We then place a total of $3 - p$ rooks in the regions marked '3'; this can be done in $\binom{9-p}{3-p} (3 - p)!$ ways. At this point, there are six rooks left to place, one in each of the six columns of the regions marked '4'. Since six rooks have already been placed, only six rows remain open. Hence there are $6!$ ways to

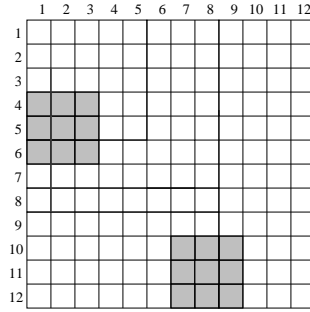


FIGURE 16. The board $B_{12}^{X,Y,Z}$.

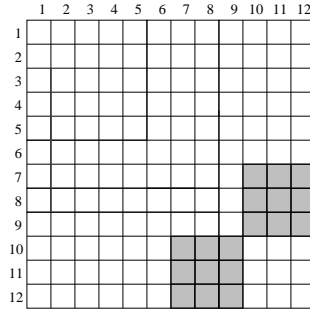


FIGURE 17. The board B' , rook-equivalent to $B_{12}^{X,Y,Z}$.

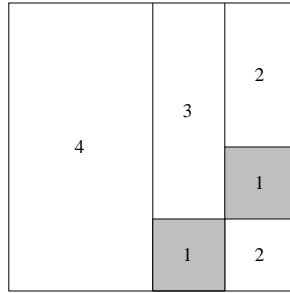


FIGURE 18. A diagram indicating the order of placement of rooks on B' .

do this last step. Thus we have

$$\begin{aligned}
 P_{12,s}^{X,Y,Z} &= \sum_{\substack{p+q=s \\ 0 \leq p \leq 3 \\ 0 \leq q \leq 3}} \binom{3}{p} \binom{3}{p} p! \cdot \binom{3}{q} \binom{3}{q} q! \cdot \binom{9-p}{3-q} (3-q)! \cdot \binom{9-p}{3-p} (3-p)! \cdot 6! \\
 &= \sum_{\substack{p+q=s \\ 0 \leq p \leq 3 \\ 0 \leq q \leq 3}} \frac{(3!)^4}{((3-p)!)^2 ((3-q)!)^2 p! q!} \frac{((9-p)!)^2}{(6-p+q)!}
 \end{aligned}$$

7. FURTHER QUESTIONS

In this section, we discuss some open questions and directions for further research.

Relations between $P_{\rho,s}^X$ and $P_{n,s}^X$. It is easy to see that for any composition $\rho = (\rho_1, \rho_2, \dots, \rho_m)$ of n , we have

$$\rho_1! \rho_2! \cdots \rho_m! |R(\rho)| = |S_n|.$$

In fact there is a natural bijection

$$\begin{aligned} \chi : (S_{\rho_1} \times S_{\rho_2} \times \cdots \times S_{\rho_m}) \times R(\rho) &\longrightarrow S_n, \\ ((\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(m)}), w) &\longmapsto \sigma, \end{aligned}$$

where σ is obtained from w by replacing the i th occurrence of 1 by $\phi_i^{(1)}$, the i th occurrence of 2 by $\rho_1 + \phi_i^{(2)}$, and so on. For example, $\chi((21, 312), 12212) = 25314$. As another example, if $\phi^{(j)}$ is the identity permutation for each j , then $\chi((\phi^{(1)}, \phi^{(2)}, \dots, \phi^{(m)}), w)$ is just the usual *standardization* of w , written $std(w)$, which is obtained from w by replacing the i th occurrence of 1 with i , the i th occurrence of 2 with $\rho_1 + i$, and so on. However, in general it is *not* true that

$$\rho_1! \rho_2! \cdots \rho_m! P_{\rho,s}^X = P_{n,s}^X, \quad (7.1)$$

for arbitrary ρ and X . For example, if $X = \mathbb{N}$ and we are counting descents without restriction, then $P_{\rho,s}^X = 0$ for $s > \sum_{i=1}^{m-1} \rho_i$, since the largest number m cannot be the bottom of a descent. On the other hand, $P_{n,s}^X$ is non-zero for all $0 \leq s \leq n-1$.

In light of the above, the following consequence of Corollaries 3.2 and 5.5 is quite surprising. For $X = 2\mathbb{N}$ and $\rho = \underbrace{(k, k, \dots, k)}_{2n}$, we have

$$(k!)^{2n} P_{\rho,s}^X = P_{2kn,s}^X.$$

This identity does not follow by applying the bijection χ . For example, $w = 41421323$ has two even descents, while $std(w) = 71832546$ has one. We therefore ask for an explicit bijection

$$\psi : \underbrace{(S_k \times S_k \times \cdots \times S_k)}_{2n} \times \underbrace{R(k, k, \dots, k)}_{2n} \longrightarrow S_{2kn},$$

satisfying

$$\overleftarrow{des}_E(\psi(w)) = \overleftarrow{des}_E(w)$$

for all $w \in R(\underbrace{k, k, \dots, k}_{2n})$. We also ask if there are other sets X and compositions ρ for which (7.1) holds.

q -analogues. The following q -analogue of the numbers $P_{n,s}^X = P_{n,s}^{X, \mathbb{N}}$ exists. Let

$$[n]_q = 1 + q + q^2 + \cdots + q^{n-1}.$$

Let Δ_{n+1}^q and Γ_{n+1}^q be the operators defined as

$$\begin{aligned} \Delta_{n+1} &: x^s \longrightarrow [s]_q x^{s-1} + q^s [n+1-s]_q x^s \\ \Gamma_{n+1} &: x^s \longrightarrow [s+1]_q x^s + q^{s+1} [n-s]_q x^{s+1}. \end{aligned}$$

Given a subset $X \subseteq \mathbb{N}$, we define the polynomials $P_n^X(q, x)$ by $P_0^X(q, x) = 1$, and

$$P_{n+1}^X(q, x) = \begin{cases} \Delta_{n+1}^q(P_n^X(q, x)), & \text{if } n+1 \notin X, \text{ and} \\ \Gamma_{n+1}^q(P_n^X(q, x)), & \text{if } n+1 \in X. \end{cases}$$

We define the coefficient polynomials $P_{n,s}^X(q)$ by setting $P_n^X(q, x) = \sum_{s \geq 0} P_{n,s}^X(q) x^s$. The polynomials $P_{n,s}^X(q)$ satisfy a recursion analogous to (2.1), and q -analogues of the formulas (2.5) and (2.6) have

been found. These results, along with a combinatorial interpretation of a Mahonian statistic *stat* satisfying

$$P_{n,s}^X(q) = \sum_{\substack{\sigma \in S_n, \\ des_X(\sigma) = s}} q^{stat(\sigma)},$$

will be presented in an upcoming paper by the authors and J. Liese.

Pattern matchings. One can put our results in a more general context of pattern matchings in permutations as follows. Given any sequence $\sigma = \sigma_1\sigma_2 \cdots \sigma_n$ of distinct integers, we let $red(\sigma)$ be the permutation that results by replacing the i -th smallest integer that appears in the sequence σ by i . For example, if $\sigma = 2\ 7\ 5\ 4$, then $red(\sigma) = 1\ 4\ 3\ 2$. Given a permutation τ in the symmetric group S_j , we define a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$ to have a τ -match at place i provided $red(\sigma_i\sigma_{i+1} \cdots \sigma_{i+j-1}) = \tau$. Let $\tau\text{-}mch(\sigma)$ be the number of τ -matches in the permutation σ . To prevent confusion, we note that a permutation not having a τ -match is different than a permutation being τ -avoiding. A permutation is called τ -avoiding if there are no indices $i_1 < i_2 < \cdots < i_j$ such that $red(\sigma_{i_1}\sigma_{i_2} \cdots \sigma_{i_j}) = \tau$. For example, if $\tau = 2\ 1\ 4\ 3$, then the permutation $3\ 2\ 1\ 4\ 6\ 5$ does not have a τ -match but it does not avoid τ since $red(2\ 1\ 6\ 5) = \tau$. In the case where $|\tau| = 2$, $\tau\text{-}mch(\sigma)$ reduces to familiar permutation statistics. That is, if $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$, let $Des(\sigma) = \{i : \sigma_i > \sigma_{i+1}\}$ and $Rise(\sigma) = \{i : \sigma_i < \sigma_{i+1}\}$. Then it is easy to see that $(2\ 1)\text{-}mch(\sigma) = des(\sigma) = |Des(\sigma)|$ and $(1\ 2)\text{-}mch(\sigma) = rise(\sigma) = |Rise(\sigma)|$. A number of recent publications have analyzed the distribution of τ -matches in permutations. See, for example, [2, 13, 12].

We can consider a more refined pattern-matching condition where we take into account conditions involving equivalence mod k for some integer $k \geq 2$. That is, suppose we fix $k \geq 2$ and we are given some sequence of distinct integers $\tau = \tau_1\tau_2 \cdots \tau_j$. Then we say that a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$ has a τ - k -equivalence match at place i provided $red(\sigma_i\sigma_{i+1} \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, 1, \dots, j-1\}$, $\sigma_{i+s} = \tau_{1+s} \pmod k$. For example, if $\tau = 1\ 2$ and $\sigma = 5\ 1\ 7\ 4\ 3\ 6\ 8\ 2$, then σ has τ -matches starting at positions 2, 5, and 6. However, if $k = 2$, then only the τ -match starting at position 5 is a τ -2-equivalence match. (Later, it will be explained that the τ -match starting at position 2 is a (1 3)-2-equivalence match and the τ -match starting a position 6 is a (2 4)-2-equivalence match.) Let $\tau\text{-}k\text{-}emch(\sigma)$ be the number of τ - k -equivalence matches in the permutation σ .

More generally, if Υ is a set of sequences of distinct integers of length j , then we say that a permutation $\sigma = \sigma_1\sigma_2 \cdots \sigma_n \in S_n$ has a Υ - k -equivalence match at place i provided that there is a $\tau \in \Upsilon$ such that $red(\sigma_i\sigma_{i+1} \cdots \sigma_{i+j-1}) = red(\tau)$ and for all $s \in \{0, \dots, j-1\}$, $\sigma_{i+s} = \tau_{1+s} \pmod k$. Let $\Upsilon\text{-}k\text{-}emch(\sigma)$ be the number of Υ - k -equivalence matches in the permutation σ .

One can then study the polynomials

$$\begin{aligned} T_{\tau,k,n}(x) &= \sum_{\sigma \in S_n} x^{\tau\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^n T_{\tau,k,n}^s x^s \text{ and} \\ U_{\Upsilon,k,n}(x) &= \sum_{\sigma \in S_n} x^{\Upsilon\text{-}k\text{-}emch(\sigma)} = \sum_{s=0}^n U_{\Upsilon,k,n}^s x^s. \end{aligned}$$

In particular, suppose that we focus on the special cases of these polynomials where we consider only patterns of length 2. That is, fix $k \geq 2$ and let A_k equal the set of all sequences $(a\ b)$ such that $1 \leq a < b \leq 2k$ and there is no lexicographically smaller sequence $x\ y$ having the property that $x \equiv a \pmod k$ and $y \equiv b \pmod k$. For example,

$$A_4 = \{1\ 2, 1\ 3, 1\ 4, 1\ 5, 2\ 3, 2\ 4, 2\ 5, 2\ 6, 3\ 4, 3\ 5, 3\ 6, 3\ 7, 4\ 5, 4\ 6, 4\ 7, 4\ 8\}.$$

Let $D_k = \{b\ a : a\ b \in A_k\}$ and $E_k = A_k \cup D_k$. Thus E_k consists of all k -equivalence patterns of length 2 that we could possibly consider. Note that if $\Upsilon = A_k$, then $\Upsilon\text{-}k\text{-}emch(\sigma) = rise(\sigma)$ and if $\Upsilon = D_k$, then $\Upsilon\text{-}k\text{-}emch(\sigma) = des(\sigma)$.

Liese [17] studied the polynomials $U_{\Upsilon,k,n}^s$ where Υ consists of patterns of length 2. For example, he showed that one can use inclusion-exclusion to find a formula for $U_{\Upsilon,k,n}^s$ for any $\Upsilon \subset E_k$ in terms of

certain rook numbers of a sequences of boards associated with Υ . The same is true for coefficients of the the polynomials $P_n^{X,Y,Z}(x)$. This approach leads to completely different formulas than the ones produced in this paper. While this approach is straightforward, it is unsatisfactory since it reduces the computation of $U_{\Upsilon,k,n}^s$ to another difficult problem, namely, computing rook numbers for general boards.

Liese [17] was able to give direct formulas for the coefficients $T_{\tau,k,n}^s$ where $\tau \in E_k$. For example, in the case where $\tau = (1\ k)$, his results imply that for all $0 \leq s \leq n$ and for all $0 \leq j \leq k - 1$,

$$T_{(1\ k),k,kn+j}^s = ((k-1)n+j)! \sum_{r=0}^s (-1)^{s-r} ((k-1)n+j+r)^n \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{s-r},$$

$$\text{and}$$

$$T_{(1\ k),k,kn+j}^s = ((k-1)n+j)! \sum_{r=0}^{n-s} (-1)^{n-s-r} (1+r)^n \binom{(k-1)n+j+r}{r} \binom{kn+j+1}{n-s-r}.$$

These two formulas are easily seen to be equivalent to special cases of our formulas. However, Liese has produced explicit formulas for $U_{\Upsilon,k,n}^s$ in the special case where Υ is a subset of the form $\{(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)\}$, where for all i, j $y_i \equiv y_j \pmod k$ and either $\Upsilon \subseteq A_k$ or $\Upsilon \subseteq D_k$. These formulas cannot always be reduced to special cases of our formulas.

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