Combinatorial Deformations of the Full Transformation Semigroup

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Contents

Chapter 1

Introduction

The representation theory of the Symmetric Group plays a large role in many areas of combinatorics, for example in the study of symmetric functions. The Full Transformation Semigroup is perhaps the most natural generalization of the Symmetric Group, and yet its representation theory is not well understood.

The objective of this work is to improve the understanding of the algebraic and combinatorial structure of the Full Transformation Semigroup. That this structure is particularly complicated is reflected in the fact that the associated Full Transformation Semigroup algebra is not semisimple. The main results follow from several attempts to introduce parameters in such a way that the algebra becomes generically semisimple. One such deformation leads to an eigenvalue result involving Schur functions (Chapter 3), while another makes the algebra "as semisimple as possible" (Chapter 4). In the final chapter we present several results pertaining to the radical of the non-deformed algebra.

1.1 The Full Transformation Semigroup T_n

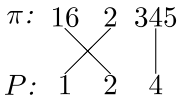
The *Full Transformation Semigroup on n letters*, denoted T_n, is the semigroup of all set maps $w : [n] \to [n]$, where $[n] = \{1, 2, \ldots, n\}$ and the multiplication is the usual composition. Such maps can be depicted in several ways; we will most often use one-line notation, for example $w = 214442$ denotes the map sending 1 to 2, 2 to 1, 3 to 4, etc.

Maps in T_n are indexed by triples (π, P, ϕ), where P is the image of the map, π is the set partition of $[n]$ whose blocks are the inverse images of the elements of P, and ϕ is the permutation describing which block is mapped to which element of the image. In what follows, $\pi = \{\pi_1, \pi_2, \ldots\}$ will always denote a set partition of $[n]$ with blocks ordered by increasing smallest element. Similarly in writing $P = \{p_1, p_2, \ldots\}$ a subset of $[n]$ we shall always intend $p_1 < p_2 < \ldots$. Permutations will be written in cycle notation.

With these conventions, we shall let $w_{\pi,P,\phi} \in T_n$ denote the map taking $x \in \pi_i$ to $p_{\phi(i)}$.

Example 1.1.1. *For* $w = 214442 \in T_6$ *we have* $\pi(w) = 16|2|345, P(w) = \{1, 2, 4\}$, *and*

$\phi(w) = (12)(3)$, *the transposition exchanging* 1 *and* 2 *and fixing* 3*. The permutation* ϕ *is most easily visualized in the following diagram of* w*.*



The invertible elements of T_n, i.e., the bijective maps, form a subsemigroup isomorphic to the Symmetric Group S_n. Thus the elements of T_n can be thought of as generalized permutations, and we can ask which of the many combinatorial aspects of the Symmetric Group can be extended in a meaningful way to the Full Transformation Semigroup.

Let $\mathbb{C}T_n$ denote the Full Transformation Semigroup algebra, consisting of complex linear combinations of elements of T_n. $\mathbb{C}T_n$ has a chain of two-sided ideals

$$\mathbb{C}T_n = I_n \supseteq I_{n-1} \supseteq \ldots \supseteq I_1 \supseteq I_0 = 0,$$

where for $1 \leq k \leq n$, I_k as a vector space is the complex span of the maps of rank less than or equal to k (the *rank* of a map is the cardinality of its image). For $1 \leq k \leq n$ define the algebras $A_{n,k} = I_k / I_{k-1}$. We can think of $A_{n,k}$ as being the algebra spanned by the maps of rank k, where two maps multiply to zero if their composition has rank less than k. The top quotient $A_{n,n}$ is isomorphic to $\mathbb{C}S_n$, the group algebra of the Symmetric Group, and is therefore semisimple. However $A_{n,k}$ is not semisimple for $k < n$, meaning that the radical $\sqrt{A_{n,k}}$ is non-trivial.

It is known that the irreducible modules for $A_{n,k}$ are indexed by partitions $\lambda \vdash k$. In fact Hewitt and Zuckerman give a calculation in [7] that generates all irreducible matrix representations for $A_{n,k}$. However, their methods are difficult to apply in practice and do not even determine the dimensions of the representations. These dimensions are known, thanks to a more recent character result of Putcha [12]. Regardless of the approach, it is clear that the non-semisimplicity of $A_{n,k}$ causes great difficulties. This has led us to define several deformations of $A_{n,k}$, with the aim of making the algebra generically semisimple.

1.2 Summary of Chapter 3

The Symmetric Group S_k acts on the maps of rank k by

$$\sigma w_{\pi,P,\phi} := w_{\pi,P,\sigma\phi}.$$

We define

$$w_1 \circ w_2 := \sum_{\sigma \in S_k} p_{\rho(\sigma)} \sigma w_1 w_2,$$

where $\rho(\sigma) \vdash k$ is the cycle type of σ, and $p_{\rho(\sigma)}$ is the corresponding power-sum symmetric function in the variables $x_1, \ldots, x_k$. Let $A_{n,k}(\mathbf{x})$ denote the algebra with the multiplication $\circ$. Explicit calculations for small values of n and k suggest that $A_{n,k}(\mathbf{x})$ is no more semisimple than $A_{n,k}$, i.e., that even for generic values of the x_i we have $\dim\sqrt{A_{n,k}(\mathbf{x})} = \dim\sqrt{A_{n,k}}$. However something interesting does come out of this multiplication.

For every finite-dimensional algebra there is an associated Gram matrix whose nullspace is the radical of the algebra. Let $M_{n,k}(\mathbf{x})$ be the (normalized) Gram matrix for $A_{n,k}(\mathbf{x})$. We use Frobenius' factorization of the group determinant (see [2]) to derive the following result about $M_{n,n}(\mathbf{x})$.

Theorem 1.2.1. *The eigenvalues of $M_{n,n}(\mathbf{x})$ are $\pm\big(\frac{n!}{f^\lambda}s_\lambda\big)^2$, $\lambda \vdash n$, where the positive values appear with multiplicity $\binom{f^\lambda+1}{2}$ and the negative values appear with multiplicity $\binom{f^\lambda}{2}$. As a consequence, the algebra $A_{n,n}(\mathbf{x})$ is semisimple if and only if the values of the parameters x_i avoid the zeros of the Schur functions s_λ.*

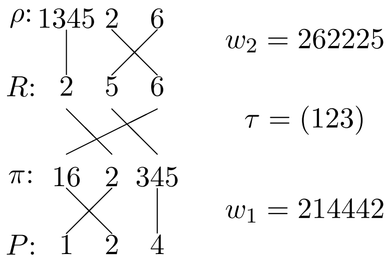
An analogous result for $k < n$ is so far only conjectural.

Conjecture 1.2.2. *The non-zero eigenvalues of $M_{n,k}(\mathbf{x})$, $k < n$, are constant multiples of the squares of the Schur functions s_λ for $\lambda \vdash k$. The multiplicity of s_λ^2 is $(\binom{n}{k}f^\lambda)^2$ for $\lambda \vdash k$, $\lambda \neq 1^k$, and $\binom{n-1}{k-1}^2$ for $\lambda = 1^k$.*

1.3 Summary of Chapter 4

There is another associative multiplication we can define on $A_{n,k}$. Let $w_1 = w_{\pi,P,\phi}$ and $w_2 = w_{\rho,R,\psi}$ be two maps of rank k. Notice that in order for the product w_1w_2 to be nonzero in $A_{n,k}$ each element of $R = P(w_2)$ must lie in a different block of $\pi = \pi(w_1)$. In this situation we can associate to the maps w_1 and w_2 the permutation $\tau \in S_k$ defined by the condition $r_i \in \pi_{\tau(i)}$.

For example, if $w_1 = 214442$ and $w_2 = 262225$ then $\tau = (123)$, as can be seen in the following diagram.



We now define a new associative multiplication in $A_{n,k}$ by

$$w_1 * w_2 := x^{\mathrm{inv}(\tau)} w_1 w_2,$$

where $\mathrm{inv}(\tau)$ is the number of inversions of τ. Let $A_{n,k}(x)$ denote the algebra with the multiplication $*$. $A_{n,k}(x)$ is not semisimple in general, however our work shows that in some sense $A_{n,k}(x)$ is "as semisimple as possible" for generic x.

The results in this Chapter are obtained by recasting $A_{n,k}$, and more generally $A_{n,k}(x)$, as *Munn matrix algebras*. These are algebras of rectangular matrices of a fixed size, with entries in a base algebra, in which multiplication is defined by inserting a fixed "sandwich matrix" Π between elements of the algebra: $X \cdot Y := X\Pi Y$.

The deformed algebra $A_{n,k}(x)$ turns out to be isomorphic to the Munn matrix algebra of $\binom{n}{k} \times S(n,k)$ matrices over $\mathbb{C}S_k$, with $S(n,k) \times \binom{n}{k}$ sandwich matrix $\Pi(x)$ defined by

$$(\Pi(x))_{\pi,P} = \begin{cases} x^{inv(\tau)}\tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \leq i \leq k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

It is natural to ask in what way the semisimplicity of a Munn matrix algebra depends on properties of the sandwich matrix Π. Clifford and Preston [3] have shown that a Munn matrix algebra is semisimple if and only if Π is "invertible", in a sense that will be made precise later. In particular semisimplicity requires that Π be a square matrix.

Clifford and Preston's result implies that $A_{n,k}(x)$ cannot be semisimple for any value of x, since the sandwich matrix $\Pi(x)$ is not square. However we can say a good deal about the "semisimple part" of $A_{n,k}(x)$, i.e., the quotient $A_{n,k}(x)/\sqrt{A_{n,k}(x)}$.

The *rank* of a matrix is the size of its largest non-singular minor. A result of McAlister [10] states that if $\mathcal{A}$ is a Munn matrix algebra, and the sandwich matrix Π is *suitable* of rank r, then $\mathcal{A}/\sqrt{\mathcal{A}} \cong \left(A/\sqrt{A}\right)_r$, the algebra of all $r \times r$ matrices with entries in $A/\sqrt{A}$. (Suitability is a technical condition that we discuss later.)

It turns out that $\Pi(x)$ has full rank $\binom{n}{k}$ for generic x. To show that the rank of $\Pi(x)$ is indeed $\binom{n}{k}$ we define a special set of partitions of $[n]$, $\binom{n}{k}$ in number, which we call *cyclically contiguous* partitions. We let $\Pi^c(x)$ be the $\binom{n}{k} \times \binom{n}{k}$ submatrix of $\Pi(x)$ consisting of the rows corresponding to the cyclically contiguous partitions. We show that $\Pi^c(x)$ is non-singular for generic x. We also verify that $\Pi(x)$ satisfies the suitability condition, and so applying McAlister's result we obtain the following theorem.

Theorem 1.3.1. *For generic x, $A_{n,k}(x)/\sqrt{A_{n,k}(x)} \cong (\mathbb{C}S_k)_{\binom{n}{k}}$.*

Note that the rank of Π cannot be any larger than its width $\binom{n}{k}$. So no deformation of $A_{n,k}$ that preserves the Munn matrix algebra structure can be any more semisimple than $A_{n,k}(x)$.

In the remainder of Chapter 4 we describe a connection between our deformed algebra $A_{n,k}(x)$ and the Rook Monoid algebra. Let $A^c_{n,k}(x)$ be the subalgebra of $A_{n,k}(x)$ spanned by the maps whose partitions are cyclically contiguous. $A^c_{n,k}(x)$ is also a Munn matrix algebra, with sandwich matrix $\Pi^c(x)$. We have found a further deformation of $A^c_{n,k}(x)$ that allows us to interpolate between $A^c_{n,k}(x)$ and a quotient of the Rook Monoid algebra. This is important, as the representation theory of the latter is is well understood (see [6]).

1.4 Summary of Chapter 5

In the final chapter we collect together several results concerning the radical of the undeformed algebra $A_{n,k}$. For example, while the associated Gram matrix $M_{n,k}$ has a complicated structure, it can be interpreted as something like the adjacency matrix of a certain graph. Using properties of this graph we are able to determine the unique maximum eigenvalue of $M_{n,k}$.

Theorem 1.4.1. *For a partition π of $[n]$ let $[[\pi]]$ denote the product of the block sizes of π. Let $\lambda = \sum_\pi [[\pi]]$, where the sum is over all partitions π of $[n]$ into k blocks such that the elements $1, 2, \ldots, k$ lie in different blocks. Then λ is the unique maximum eigenvalue for the normalized Gram matrix $M_{n,k}$. The corresponding eigenvector is $\mathbf{v} = \sum\limits_{w \in T_{n,k}} [[\pi(w)]]w$.*

In another result we verify that two characterizations of the radical of $A_{n,k}$, one as the nullspace of the associated Gram matrix, the other in terms of the sandwich matrix, do indeed give the same equations on the matrix entries.

One of our original questions about the non-deformed algebras $A_{n,k}$ was whether we could provide some sort of combinatorial description for the radical $\sqrt{A_{n,k}}$. A basis might be too much to hope for, but even a spanning set would be very useful. We have found several classes of elements in $\sqrt{A_{n,k}}$, for example the class defined in the following result.

Proposition 1.4.2. *Fix a subset $P = \{p_1, \ldots, p_k\} \subseteq [n]$ and an integer partition $\lambda = (\lambda_1, \ldots, \lambda_k)$ of n into distinct parts, $k > 1$. Given a set partition $\pi = \{\pi_1, \ldots, \pi_k\}$ of type λ we associate a permutation ν_π by $|\pi_i| = \lambda_{\nu_\pi(i)}$. Then*

$$x_{\lambda,P} := \sum_\pi sgn(\nu_\pi) \sum_{\sigma \in S_k} sgn(\sigma) w_{\pi,P,\sigma} \in \sqrt{A_{n,k}},$$

where the outer sum is over all set partitions π of type λ.

These elements and the others we describe do not in general span $\sqrt{A_{n,1}}$, nor are they independent.

Finally, we describe two interesting actions of S_n on $\sqrt{A_{n,k}}$, one by conjugation and the other by left multiplication. We give a character result for the case $k = 1$.

Chapter 2

Preliminaries

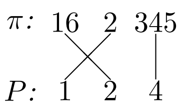
2.1 The Full Transformation Semigroup T_n

In this chapter we introduce structures and techniques that will play a central role in our main results. We begin with our "main character", the Full Transformation Semigroup.

Definition 2.1.1. *A* semigroup S *is a set together with a multiplication* $\mu : S \times S \longrightarrow S$*. The* Full Transformation Semigroup *is the semigroup* T_n *of all set maps* $w : [n] \to [n]$*, where maps multiply by composition. We shall denote such a map* w *by* $w = w_{\pi,P,\phi}$*, where* P *is the image,* π *is the set partition whose blocks are the inverse images of the elements of* P*, and* ϕ *is determined by the rule* $w_{\pi,P,\phi}(\pi_i) = p_{\phi(i)}$*.*

We repeat here the example from the introduction.

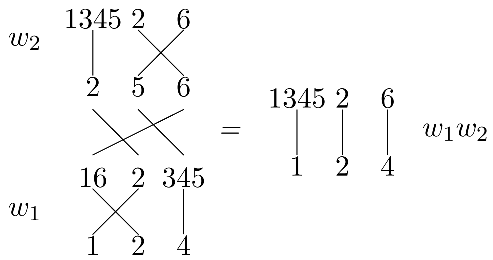
Example 2.1.2. *For* $w = 214442$ *we have* $\pi(w) = 16|2|345, P(w) = \{1, 2, 4\}$*, and* $\phi(w) = (12)(3)$*, as in the diagram below.*



Note that for $\phi(w)$ to be well defined we need to canonically order the elements of the image and the blocks of the partition. As mentioned in the introduction the elements of $P = \{p_1, p_2, \ldots\}$ will always be labeled in increasing order, and the blocks of $\pi = \{\pi_1, \pi_2, \ldots\}$ will always be labeled in increasing order of smallest element. We shall let $\binom{[n]}{k}$ denote the set of k-subsets of $[n]$. and $S([n], k)$ denote the set of partitions of $[n]$ into k blocks.

The composition of two maps is best understood with a diagram, as in the following example.

Example 2.1.3. *Let* $w_1 = 214442$ *and* $w_2 = 262225$*. Then* $w_1 w_2 = 121114$*.*



One reason for the importance of the Full Transformation Semigroup can be seen in the following result [3], which is an analogue of Cayley's Theorem for groups.

Proposition 2.1.4. *Every finite (associative) semigroup is a subsemigroup of T_n, for some n.*

Proof. Suppose S is a finite associative semigroup with $|S| = n$. Label the elements of S as $s_1, s_2, \ldots, s_n$. Then identify $s \in S$ with the map in T_n taking s_i to $s_j = ss_i$. $\square$

Remark 2.1.5. *Elements of T_n can be thought of as $n \times n$ (0,1)-matrices with exactly one 1 in each column, for example*

$$214442 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In this context, composition of maps is realized by multiplication of the corresponding matrices.

The *rank* of a map is the cardinality of its image, In the sequel it will often be useful to restrict ourselves to maps of a fixed rank.

Definition 2.1.6.

$$\begin{aligned} T_{n,k} &:= \{w \in T_n | \text{rank}(w) = k\} \\ &= \{w_{\pi,P,\phi} | \pi \in S([n], k), P \in \binom{[n]}{k}, \phi \in S_k\} \end{aligned}$$

As special cases, we have

$$\begin{aligned} T_{n,n} &\cong S_n, \text{ the Symmetric Group, and} \\ T_{n,1} &\cong C_n, \end{aligned}$$

where C_n denotes the n-element semigroup satisfying $xy = x$ for all $x, y \in C_n$. In general, we have

$$\begin{aligned} |T_{n,k}| &= S(n,k)\binom{n}{k}k!, \text{ and} \\ |T_n| &= \sum_{k=1}^{n} |T_{n,k}| = \sum_{k=1}^{n} S(n,k)\binom{n}{k}k! = n^n. \end{aligned}$$

Remark 2.1.7. *As defined, $T_{n,k}$ is not a semigroup for $1 < k < n$ as it is not closed under multiplication. However, we can make it into a semigroup $T^0_{n,k}$ by adjoining a zero element, and specifying that $w_1w_2 = 0$ if $\text{rank}(w_1w_2) < k$ in T_n. At times we will abuse notation and use $T_{n,k}$ when we really mean $T^0_{n,k}$. It will be clear from the context which meaning is intended.*

Recall that a (two-sided) *ideal* I in a semigroup S is a subset $I \subset S$ satisfying $si, is \in I$ for all $i \in I, s \in S$. The following proposition follows from elementary linear algebra, using the interpretation of maps as (0,1)-matrices.

Proposition 2.1.8. $I_k := \bigcup_{i=0}^{k} T_{n,i} = \{w \in T_n | \text{rank}(w) \leq k\}$ *is an ideal of* T_n

A semigroup S is called "0-simple" if $S^2 \neq 0$ and (0) is the only proper ideal of S.

Fact 2.1.9. *T_n has a composition series:*

$$T_n = I_n \supset I_{n-1} \supset \ldots \supset I_1 = \{\textit{constant maps}\},$$

where I_k/I_{k-1} is a 0-simple semigroup, for all $k = 2, \ldots, n$.

Note that $I_k/I_{k-1} \cong T^0_{n,k}$. In what follows we will focus mainly on these factor semigroups, rather than the entire semigroup T_n.

2.2 The Rook Monoid R_n

Although the main focus of this is the Full Transformation Semigroup, at times it will be instructive to consider another semigroup, known as the Rook Monoid, which also generalizes the Symmetric Group.

Definition 2.2.1. *The* Rook Monoid *(also know as the* Symmetric Inverse Semigroup*) is the semigroup R_n of all partial bijections of $[n]$, where a partial bijection consists of a domain S and an image T, both subsets of $[n]$, and a bijection between S and T. In addition, the Rook Monoid contains the "empty map", denoted 0, with $S = T = \emptyset$. The* rank *of a partial bijection is the common size of its domain and image, and we shall denote partial bijections by $w_{S,T,\sigma}$, where S is the domain, T is the image, and σ is determined by the rule $w_{S,T,\sigma}(s_i) = t_{\sigma(i)}$.*

Example 2.2.2. *For* $w = \begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & 2 \end{pmatrix}$ *we have* $S(w) = \{2, 4, 5\}, T(w) = \{1, 2, 6\}$*, and* $\sigma(w) = (132)$

Partial bijections multiply by composition, as in the following two examples.

Example 2.2.3.

$$\begin{pmatrix} 1 & 4 & 5 & 6 \\ 4 & 3 & 1 & 6 \end{pmatrix} \begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 4 \\ 6 & 4 \end{pmatrix}$$

$$\begin{pmatrix} 3 & 4 & 5 \\ 1 & 2 & 4 \end{pmatrix} \begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & 2 \end{pmatrix} = 0$$

(In general, $w_1w_2 = 0$ if $T(w_2) \cap S(w_1) = \emptyset$.)

Remark 2.2.4. *Elements of R_n can be thought of as $n \times n$ (0,1)-matrices with at most one 1 in each row and column, for example*

$$\begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

As with T_n, composition of partial bijections is realized by multiplication of the corresponding matrices.

In analogy with the Full Transformation Semigroup we shall often want to restrict ourselves to partial bijections of a fixed rank.

Definition 2.2.5.

$$\begin{aligned} R_{n,k} &:= \{w \in R_n | rank(w) = k\} \\ &= \{w_{S,T,\sigma} | S, T \in \binom{[n]}{k}, \sigma \in S_k\} \end{aligned}$$

As special cases, we have

$$\begin{aligned} R_{n,n} &= S_n, \text{ the Symmetric Group, and} \\ R_{n,0} &= \{0\}, \text{ the empty map.} \end{aligned}$$

In general, we have

$$\begin{aligned} |R_{n,k}| &= \binom{n}{k}^2 k! \\ |R_n| &= \sum_{k=0}^{n} |R_{n,k}| = \sum_{k=0}^{n} \binom{n}{k}^2 k! \end{aligned}$$

The algebraic structure of the Rook Monoid is simpler than that of the Full Transformation Semigroup, a circumstance deriving from the fact that each partial bijection has a well-defined inverse.

Definition 2.2.6. *For S a semigroup, $a, b \in S$ are* inverses *if $aba = a$ and $bab = b$. A semigroup S is an* inverse semigroup *if every element of S has a unique inverse.*

Proposition 2.2.7. *R_n is an inverse semigroup.*

Proof. We show that the unique inverse of $a = w_{S,T,\sigma}$ is $b = w_{T,S,\sigma^{-1}}$. Certainly $aba = a$ and $bab = b$. On the other hand, if $c = w_{P,Q,\phi}$ then $aca = a$ and $cac = c$ taken together imply that $\text{rank}(a) = \text{rank}(c)$. For the rank to be preserved in the composition aca we need $P = T$ and $Q = S$. Finally, $aca = w_{S,T,\sigma} w_{T,S,\phi} w_{S,T,\sigma} = w_{S,T,\sigma\phi\sigma} = w_{S,T,\sigma}$, and so $\phi = \sigma^{-1}$. □

Note that if $a = w_{S,T,\sigma}$ and $b = w_{T,S,\sigma^{-1}}$ we have $ab = w_{S,S,\mathrm{id}} = I_S$, the identity map on S. Likewise, we have $ba = I_T$.

Example 2.2.8. *If* $w = \begin{pmatrix} 2 & 4 & 5 \\ 6 & 1 & 2 \end{pmatrix}$, *then* $w^{-1} = \begin{pmatrix} 1 & 2 & 6 \\ 4 & 5 & 2 \end{pmatrix}$, *and*

$$ww^{-1} = \begin{pmatrix} 1 & 2 & 6 \\ 1 & 2 & 6 \end{pmatrix} = I_{\{1,2,6\}}, \text{ while } w^{-1}w = \begin{pmatrix} 2 & 4 & 5 \\ 2 & 4 & 5 \end{pmatrix} = I_{\{2,4,5\}}.$$

The following result of Vagner and Preston ([3], Section 1.9) is analogous to Proposition 2.1.4.

Proposition 2.2.9. *Every inverse semigroup can be embedded in* R_N *for some* N.

As with T_n the notion of rank allows us to define a composition series for R_n.

Fact 2.2.10. R_n *has a composition series:*

$$R_n = J_n \supset J_{n-1} \supset \ldots \supset J_1 \supset J_0 = 0$$

where $J_k := \bigcup_{i=0}^{k} R_{n,i} = \{w \in R_n | \mathrm{rank}(w) \leq k\}$ *is an ideal of* R_n, *and* J_k/J_{k-1} *is a* 0*-simple inverse semigroup for each* $k = 1, \ldots, n$.

2.3 Combinatorial Representation Theory

In what follows we will briefly describe several combinatorial questions that often arise in the study of algebras, and give the answers in a few classical cases. Barcelo and Ram give an excellent overview of the topic in [1]; we shall briefly summarize their approach. As elsewhere in this work, all algebras are finite-dimensional, associative, and over $\mathbb{C}$.

Recall that an A-module M over an algebra A is a vector space with an associated linear A-action.

Definition 2.3.1. *An* A*-module* M *is* irreducible *if it has no non-trivial proper submodules.* M *is* indecomposable *if it cannot be written as the direct sum of two proper submodules. Irreducible modules are by definition indecomposable, but in general the converse is false.*

Definition 2.3.2. *An algebra* A *is* semisimple *if the only indecomposable modules are the irreducible ones. Equivalently, an algebra is semisimple if and only if it has no non-zero nilpotent ideals.*

Definition 2.3.3. *The* radical $\sqrt{A}$ *of* A *is the maximal nilpotent ideal of* A. *Equivalently,* $\sqrt{A}$ *is the minimal ideal* I *such that* A/I *is semisimple.*

Semisimple algebras are reasonably well understood. By Wedderburn's Theorem (see [4], Section 2.4) every semisimple algebra over $\mathbb{C}$ is isomorphic to a direct product of matrix algebras over $\mathbb{C}$. The irreducible modules of a semisimple algebra are in one-to-one correspondence with these irreducible components of the algebra. On the other hand, a non-semisimple algebra may have infinitely many isomorphism classes of indecomposable modules!

Given a group or an algebra, we can ask the following questions.

- Is there a nice indexing set for the irreducible modules?
- What are their dimensions?
- Is there a combinatorial formula for the irreducible characters?
- Is there a combinatorial construction of the irreducible modules?

The answers to these questions are "yes" in several classical cases, for example the Symmetric Group and the General Linear Group. The answers are also "yes" for the Rook Monoid, as we shall see shortly. For the Full Transformation Semigroup the first three questions have been answered in the affirmative (see [11],[12]). It is one of our goals to answer the fourth.

Example 2.3.4. The Symmetric Group S_n: *The irreducible modules of S_n are in bijection with partitions $\lambda \vdash n$. The dimension of the irreducible module S^λ is given by the hook length formula $f^\lambda = \frac{n!}{\prod_{x\in\lambda} h_x}$, which also counts the number of standard Young tableaux of shape λ. The irreducible characters are given as a sum over standard Young tableaux: $\chi^\lambda(\mu) = \sum_T wt^\mu(T)$. The irreducible modules are realized combinatorially as the* Specht modules. *James [8], Sagan [13], and many others, have given a detailed treatment of the representation theory of the Symmetric Group.*

Example 2.3.5. The Rook Monoid R_n: *A theorem of Clifford and Preston [3] states that if S is an inverse semigroup, then $\mathbb{C}S$ is semisimple. As a consequence, $\mathbb{C}R_n$ is semisimple. The irreducible modules for $\mathbb{C}R_n$ are indexed by partitions $\lambda \vdash k$, for $k = 0, 1, \ldots, n$. For $\lambda \vdash k$, the corresponding irreducible module M^λ has dimension $\binom{n}{k} f^\lambda$. Munn [11] has given a formula for the irreducible characters that can be restated so that it is analogous to that of the characters of the Symmetric Group. Grood [6] has given a combinatorial construction of the irreducible characters which mirrors that of the Specht modules for the Symmetric Group.*

Non-semisimple algebras are in general much more difficult to understand. One common source of non-semisimple algebras is *modular representation theory*, which is representation theory over a field of positive characteristic. This area is beyond the scope of our work; however before moving on we give a simple example of a non-semisimple algebra.

Example 2.3.6. The algebra UT_n of $n \times n$ upper triangular matrices: *The algebra UT_n is not semisimple, and indeed the radical is easy to describe. $\sqrt{UT_n}$ is the ideal consisting of the $n \times n$* strictly *upper triangular matrices.*

2.4 Gram matrices

As defined, the radical of an algebra A is the maximal nilpotent ideal of A. The radical can also be characterized as the set of *strongly nilpotent* elements (see [4], Section 2.2), elements whose appearance a prescribed number of times in any product makes that product zero. Unfortunately, neither of these characterizations allows us (or a computer) to easily test whether a particular element is in the radical. The following useful result (see [5]) gives us a way to compute the radical of any finite-dimensional algebra.

Proposition 2.4.1. *Let A be a finite-dimensional algebra over $\mathbb{C}$, and $B = \{v_1, \ldots, v_n\}$ a basis for A. Identify A with $\mathbb{C}^n$ via the basis B. Define the $n \times n$ Gram matrix M by $(M)_{i,j} := \mathrm{tr}(v_i v_j)$. Then*

$$\sqrt{A} = \mathrm{null}(M).$$

To prove this proposition we first establish several lemmas.

Lemma 2.4.2. *Let $\lambda_1, \lambda_2, \ldots, \lambda_n \in \mathbb{C}$. If $\sum_{i=1}^{n} \lambda_i^k = 0$ for all $k = 1, 2, \ldots$, then $\lambda_1 = \lambda_2 = \ldots = \lambda_n = 0$.*

Proof. First of all, we show that

$$\log \prod_{i=1}^{n} \frac{1}{1 - x_i t} = \sum_{k \geq 1} \frac{1}{k} p_k(x_1, \ldots, x_n) t^k.$$

This is a simple calculation:

$$\begin{aligned}
\log \prod_{i=1}^{n} \frac{1}{1 - x_i t} &= \sum_{i=1}^{n} \log \frac{1}{1 - x_i t} \\
&= \sum_{i=1}^{n} \sum_{k \geq 1} \frac{1}{k} x_i^k t^k \qquad \text{(power series expansion of } \log \frac{1}{1-u}) \\
&= \sum_{k \geq 1} \frac{1}{k} \left(\sum_{i=1}^{n} x_i^k \right) t^k \\
&= \sum_{k \geq 1} \frac{1}{k} p_k(x_1, \ldots, x_n) t^k.
\end{aligned}$$

By assumption, $\sum_{i=1}^{n} \lambda_i^k = p_k(\lambda_1, \ldots, \lambda_n) = 0, \forall k \geq 1$, so we have $\log \prod_{i=1}^{n} \frac{1}{1-\lambda_i t} = 0$, i.e., $\prod_{i=1}^{n} \frac{1}{1-\lambda_i t} = 1$. Thus $\sum_{k \geq 0} e_k(\lambda_1, \ldots, \lambda_n) t^n = \prod_{i=1}^{n} (1 - \lambda_i t) = 1$, where e_k is the k^{th} elementary symmetric function, and so $e_k(\lambda_1, \ldots, \lambda_n) = 0$ for all $k \geq 1$. In particular, $e_n(\lambda_1, \ldots, \lambda_n) = \lambda_1 \cdots \lambda_n = 0$, so there exists a $j, 1 \leq j \leq n$ such that $\lambda_j = 0$. Without loss of generality we can take $j = n$.

Now we have $\lambda_1 \cdots \lambda_{n-1} \in \mathbb{C}$ with $\sum_{i=1}^{n-1} \lambda_i^k = 0$ for all $k = 1, 2, \ldots$. By induction, we can conclude $\lambda_1 = ... = \lambda_{n-1} = 0$. □

Lemma 2.4.3.

$$\sqrt{A} = \{a \in A | \mathrm{tr}(ax) = 0 \;\; \forall x \in A\}$$

Proof. Let $J = \{a \in A | \text{tr}(ax) = 0 \ \forall x \in A\}$, and let I be a nilpotent ideal of A. To show $J = \sqrt{A}$ we need to show that $I \subseteq J$, and that J is a nilpotent ideal.
Proof that $I \subseteq J$:

$$\begin{aligned} a \in I &\implies ax \in I \ \forall x \in A \\ &\implies ax \text{ is nilpotent } \forall x \in A \\ &\implies \text{tr}(ax) = 0 \ \forall x \in A \\ &\implies a \in J \end{aligned}$$

Proof that J is a nilpotent ideal:

$$\begin{aligned} a \in J &\implies \text{tr}(a^k) = 0 \ \forall k = 1, 2, \ldots \\ &\implies \sum_{\lambda \text{ an e-value of } a} \lambda^k = 0 \ \forall k = 1, 2, \ldots \\ &\implies \text{all eigenvalues of } a \text{ are } 0 \qquad \text{(by Lemma 2.4.2)} \\ &\implies a \text{ has Jordan form that is strictly upper-triangular} \\ &\implies a^m = 0 \text{ for some } m, \text{ i.e., } a \text{ is nilpotent} \end{aligned}$$

So J consists of nilpotent elements, but is J an ideal? Let $a \in J, x \in A$. We need to show $ax, xa \in J$. Let $y \in A$. $\text{tr}((ax)y) = \text{tr}(a(xy)) = 0$ by definition of J. Similarly, $\text{tr}((xa)y) = \text{tr}(x(ay)) = \text{tr}((ay)x) = \text{tr}(a(yx)) = 0$. So J is an ideal, which completes the proof of the second lemma. □

Proof. (of Proposition 2.4.1) Writing $a \in A$ as $a = \sum_{i=1}^{n} c_i v_i$, and identifying a with the column vector $c = [c_1, \ldots, c_n] \in \mathbb{C}^n$ we have

$$\begin{aligned} a \in \sqrt{A} &\iff tr(ax) = 0 \ \forall x \in A \\ &\iff tr(av_j) = 0, 1 \leq j \leq n \\ &\iff tr\left(\sum_{i=1}^{n} c_i v_i v_j\right) = 0, 1 \leq j \leq n \\ &\iff \sum_{i=1}^{n} tr(v_i v_j) c_i = 0, 1 \leq j \leq n \\ &\iff (Ma)_j = 0, 1 \leq j \leq n \\ &\iff Ma = 0 \\ &\iff a \in Null(M) \end{aligned}$$

□

Now we consider several examples.

2.4.1 The full matrix algebra M_n

For a basis take $\{E_{i,j} | 1 \leq i, j \leq n\}$, where $E_{i,j}$ is the $n \times n$ matrix with a 1 in the (i,j) position and zeros elsewhere. If we order the $E_{i,j}$ by $E_{1,1}, E_{2,2}, \ldots, E_{n,n}, E_{1,2}, E_{2,1}, \ldots, E_{n-1,n}, E_{n,n-1}$, i.e., first the $E_{i,i}$, then the pairs $E_{i,j}, E_{j,i}$, we have

$$M = n \begin{pmatrix} 1 & & & & & & & & & & & \\ & 1 & & & & & & & & & & \\ & & \ddots & & & & & & & & & \\ & & & 1 & & & & & & & & \\ & & & & 1 & 0 & & & & & & \\ & & & & 0 & 1 & & & & & & \\ & & & & & & 1 & 0 & & & & \\ & & & & & & 0 & 1 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & & 1 & 0 \\ & & & & & & & & & & 0 & 1 \end{pmatrix}.$$

For eigenvalues and eigenvectors we have

$$\begin{aligned} ME_{i,i} &= nE_{i,i}, 1 \leq i \leq n \\ M(E_{i,j} + E_{j,i}) &= n(E_{i,j} + E_{j,i}), 1 \leq i < j \leq n \\ M(E_{i,j} - E_{j,i}) &= -n(E_{i,j} - E_{j,i}), 1 \leq i < j \leq n. \end{aligned}$$

So the eigenvalue n has multiplicity $n + \binom{n}{2}$, while $-n$ has multiplicity $\binom{n}{2}$.

2.4.2 The algebra UT_n of upper triangular matrices

For a basis take $\{E_{i,j} | 1 \leq i \leq j \leq n\}$. Using the same ordering as above we have

$$M = \begin{pmatrix} n & & & & & & \\ & n-1 & & & & & \\ & & \ddots & & & & \\ & & & 1 & & & \\ & & & & 0 & & \\ & & & & & \ddots & \\ & & & & & & 0 \end{pmatrix}.$$

For eigenvalues and eigenvectors we have

$$\begin{aligned} ME_{i,i} &= (n+1-i)E_{i,i}, 1 \leq i \leq n \\ ME_{i,j} &= 0, 1 \leq i < j \leq n. \end{aligned}$$

Every nonzero eigenvalue has multiplicity 1, and the radical has dimension $\binom{n}{2}$, as we saw in Section 2.3.2.

We also consider certain subalgebras of UT_n.

Definition 2.4.4. *For λ a partition satisfying $\lambda_i \leq n+1-i$ for all i, i.e., for λ fitting inside the staircase shape $n, n-1, \ldots, 1$, there is a subalgebra (actually an ideal) A_λ of UT_n consisting of those matrices whose support is contained in the shape λ.*

For example, if $n = 5$ and $\lambda = 5331$, then

$$A_\lambda = \left\{ \begin{pmatrix} * & * & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & * & * & * \\ 0 & 0 & 0 & 0 & * \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \right\}.$$

In general, A_λ has basis $\{E_{i,j} | 1 \leq i \leq n, n+1-\lambda_i \leq j \leq n\}$. As with UT_n, the only elements with non-zero trace are the $E_{i,i}$, as everything else pushes nonzero entries up and to the right. $E_{i,i}$ fixes precisely the $E_{i,j}$ with $j \geq i$, so $\operatorname{tr}(E_{i,i}) = n+1-i$. In the multiplication table of the basis elements $E_{i,i}$ only appears once. This is because $E_{i,j}E_{k,l} = \delta_{j,k}E_{i,l}$ and since we are working with upper-triangular matrices, $i = l$ and $j = k$ taken together imply $i = j = k = l$.

So M has eigenvalues λ_i if $\lambda_i = n+1-i, 1 \leq i \leq n$, and the remaining eigenvalues are 0. As with UT_n, each nonzero eigenvalue has multiplicity one.

In the example above,

$$M = \begin{pmatrix} 5 & & & & \\ & 3 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{pmatrix}.$$

2.4.3 Group algebras

For G a finite group, $g \in G$ has trace equal to the number of $h \in G$ such that $gh = h$. So $\operatorname{tr}(g) = \begin{cases} 0 & g \neq \mathrm{id} \\ |G| & g = \mathrm{id} \end{cases}$, and thus $\operatorname{tr}(g_1 g_2) = \begin{cases} 0 & g_2 \neq g_1^{-1} \\ |G| & g_2 = g_1^{-1} \end{cases}$

If we order the group elements so that the involutions come first, followed by pairs

(g, g^{-1}), we have

$$M = |G| \begin{pmatrix} 1 & & & & & & & & & & & & \\ & 1 & & & & & & & & & & & \\ & & \ddots & & & & & & & & & & \\ & & & 1 & & & & & & & & & \\ & & & & 1 & 0 & & & & & & & \\ & & & & 0 & 1 & & & & & & & \\ & & & & & & 1 & 0 & & & & & \\ & & & & & & 0 & 1 & & & & & \\ & & & & & & & & \ddots & & & & \\ & & & & & & & & & 1 & 0 \\ & & & & & & & & & 0 & 1 \end{pmatrix}.$$

For eigenvalues and eigenvectors we have

$$\begin{aligned} Mg &= |G|g, g \text{ an involution} \\ M(g+g^{-1}) &= |G|(g+g^{-1}), g \text{ not an involution} \\ M(g-g^{-1}) &= -|G|(g-g^{-1}), g \text{ not an involution} \end{aligned}$$

So the eigenvalue $|G|$ has multiplicity $r+s$, where r is the number of involutions of G and $r+2s = |G|$. The eigenvalue $-|G|$ has multiplicity s.

2.4.4 The Full Transformation Semigroup T_n

Let $M_{n,k}$ be the Gram matrix associated to the algebra $A_{n,k}$.

Proposition 2.4.5.

$$(M_{n,k})_{i,j} = \begin{cases} S(n,k)k! & \textit{if } w_i w_j \textit{ has } k \textit{ fixed points, and} \\ 0 & \textit{otherwise.} \end{cases}$$

For example, ordering the rows and columns lexicographically, we have

$$
M_{3,2} = \begin{pmatrix}
0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 \\
0 & 0 & 6 & 6 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 6 \\
6 & 6 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 6 & 6 & 0 & 0 \\
0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 6 & 0 & 0 \\
6 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 6 \\
6 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 6 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 6 & 0 & 6 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 \\
0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 6 & 0 & 6 & 6 \\
0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 0 & 0 & 0 & 0 & 0 & 6 & 6
\end{pmatrix}
$$

Proof.

$$
\begin{aligned}
(M_{n,k})_{i,j} \neq 0 \quad & \text{iff} \quad \operatorname{tr}(w_i w_j) \neq 0 \\
& \text{iff} \quad \exists r \text{ s.t. } (w_i w_j)_{r,r} \neq 0 \qquad \text{(since the entries of } w_i w_j \text{ are nonnegative)} \\
& \text{iff} \quad \exists r \text{ s.t. } \sum_{s=1}^{|T_{n,r}|} (w_i)_{r,s}(w_j)_{s,r} \neq 0 \\
& \text{iff} \quad \exists r, s \text{ s.t. } (w_i)_{r,s}(w_j)_{s,r} \neq 0 \\
& \text{iff} \quad \exists r, s \text{ s.t. } (w_i)_{r,s} \neq 0 \text{ and } (w_j)_{s,r} \neq 0
\end{aligned}
$$

Now $(w_i)_{r,s} = \begin{cases} 1 & w_i w_s = w_r \\ 0 & \text{otherwise} \end{cases}$ and $(w_j)_{s,r} = \begin{cases} 1 & w_j w_r = w_s \\ 0 & \text{otherwise} \end{cases}$.

So $(M_{n,k})_{i,j} \neq 0$ if and only if there exists r and s such that

$$
\begin{aligned}
w_i w_s &= w_r, \text{ and} \\
w_j w_r &= w_s.
\end{aligned}
$$

In this case

$$
\begin{aligned}
w_i w_j w_r &= w_i w_s = w_r, \text{ and} \\
w_j w_i w_s &= w_j w_r = w_s.
\end{aligned}
$$

So $w_i w_j$ fixes w_r and $w_j w_i$ fixes w_s. But an element $w \in T_{n,k}$ fixes some other element $w' \in T_{n,k}$ if and only if w fixes all k elements of the image of w'.

So $(M_{n,k})_{i,j} \neq 0$ if and only if $w_i w_j$ and $w_j w_i$ both have k fixed points. Note here that one implies the other. Suppose $\operatorname{Im}(w_i) = \{s_1, s_2, \ldots, s_k\}$ and $\operatorname{Im}(w_j) = \{t_1, t_2, \ldots, t_k\}$.

If $w_i w_j$ fixes $s_1, s_2, \ldots, s_k$, then w_j must send s_m into $w_i^{-1}(s_m)$ for each $m = 1, 2, \ldots, k$. But then $w_j w_i(t_m) = w_j(s_m) = t_m$ for all $m = 1, 2, \ldots, k$.

Now, given such w_i and w_j,

$$(M_{n,k})_{i,j} = \#(w_r, w_s) \text{ s.t. } w_i w_j w_r = w_r \text{ and } w_j w_i w_s = w_s$$

w_s is determined by w_r, since $w_s = w_j w_r$, so

$$\begin{aligned}(M_{n,k})_{i,j} &= \#w_r \text{ s.t. } w_i w_j w_r = w_r \\ &= \#w \in T_{n,k} \text{ s.t. } \operatorname{Im}(w) = \operatorname{Im}(w_i) \\ &= S(n,k)k!\end{aligned}$$

□

From now on we will always rescale the Gram matrix for $A_{n,k}$ by dividing by $S(n,k)k!$. Data for the rescaled matrix is given in the following table.

n	k	eigenvalues
2	2	1^2
2	1	$0^1, 2^1$
3	3	$-1^1, 1^5$
3	2	$-3^1, -2^2, -1^1, 0^5, 1^3, 2^2, 3^3, 4^1$
3	1	$0^2, 3^1$
4	4	$-1^7, 1^{17}$
4	3	$-5^1, -4^3, -\sqrt{15}^6, -\sqrt{12}^3, -3^1, -\sqrt{5}^9, -2^3,$ $-1^3, 0^{55}, 1^6, 2^6, \sqrt{5}^9, 3^3, 4^6, \sqrt{12}^3, \sqrt{15}^6, 5^6, 6^1$
4	2	$-8^3, -\sqrt{28}^5, -2^{10}, 0^{39}, 2^{15}, \sqrt{28}^5, 8^6, 14^1$
4	1	$0^3, 4^1$

2.4.5 The Rook Monoid R_n

Let $B_{n,k}$ be the semigroup algebra $\mathbb{C}R_{n,k}$. For a basis take $\{w_{S,T,\sigma} | S, T \in \binom{[n]}{k}, \sigma \in S_k\}$, where $w_{S,T,\sigma}(s_i) = t_{\sigma(i)}$. Recall that

$$w_{S,T,\sigma} w_{A,B,\phi} = \begin{cases} w_{A,T,\sigma\phi} & \text{if } B = S \\ 0 & \text{otherwise.} \end{cases}$$

$w_{A,T,\sigma\phi}$ has non-zero trace if and only if $A = T$ and $\sigma\phi = \text{id}$, i.e., $\phi = \sigma^{-1}$, in which case we have $\text{tr}(w_{T,T,\text{id}}) = \text{tr}(I_T) = \binom{n}{k}k!$.

So

$$\text{tr}(w_{S,T,\sigma}w_{A,B,\phi}) = \begin{cases} \binom{n}{k}k! & \text{if } B = S, A = T, \phi = \sigma^{-1}, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

If we order the elements of $R_{n,k}$ so that the involutions come first, then pair each non-involution with its unique inverse we have

$$M = \binom{n}{k}k! \begin{pmatrix} 1 & & & & & & & & & & & \\ & 1 & & & & & & & & & & \\ & & \ddots & & & & & & & & & \\ & & & 1 & & & & & & & & \\ & & & & 1 & 0 & & & & & & \\ & & & & 0 & 1 & & & & & & \\ & & & & & & 1 & 0 & & & & \\ & & & & & & 0 & 1 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & 1 & 0 \\ & & & & & & & & & 0 & 1 \end{pmatrix}.$$

For eigenvalues and eigenvectors we have

$$\begin{aligned} Mw_{S,S,\sigma} &= \binom{n}{k}k!w_{S,S,\sigma}, \sigma \text{ an involution} \\ M(w_{S,T,\sigma} + w_{T,S,\sigma^{-1}}) &= \binom{n}{k}k!(w_{S,T,\sigma} + w_{T,S,\sigma^{-1}}) \\ M(w_{S,T,\sigma} - w_{T,S,\sigma^{-1}}) &= -\binom{n}{k}k!(w_{S,T,\sigma} - w_{T,S,\sigma^{-1}}), \end{aligned}$$

where in the last two cases we permit $T = S$ so long as σ is not an involution.

So the eigenvalue $\binom{n}{k}k!$ has multiplicity $\binom{n}{k}r + \frac{1}{2}(\binom{n}{k}2s + (\binom{n}{k}^2 - \binom{n}{k})k!)$, where $r = \sum_{\lambda\vdash k} f^\lambda$ is the number of involutions of S_k and $r + 2s = k!$, while $-\binom{n}{k}k!$ has multiplicity $\frac{1}{2}(\binom{n}{k}2s + (\binom{n}{k}^2 - \binom{n}{k})k!)$.

2.5 Rees matrix semigroups and Munn matrix algebras

Given a group G let G^0 denote the semigroup formed by adjoining a zero element to G, satisfying $0 \cdot g = g \cdot 0 = 0$ for all $g \in G$. G^0 is called a *group with zero.*

Definition 2.5.1. *The* Rees matrix semigroup $\mathcal{S} = \mathcal{M}(G^0; m, n; \Pi)$ *consists of the set of all $m \times n$ matrices over G^0 having a single non-zero entry. Π is an $n \times m$ matrix over G^0, called the* sandwich matrix. *Multiplication in $\mathcal{S}$ is defined by $X \cdot Y := X\Pi Y$.*

Fact 2.5.2. *$T_{n,k}$ is isomorphic to the Rees matrix semigroup $\mathcal{M}(S_k^0; \binom{[n]}{k}, S([n],k); \Pi_{n,k})$. The elements of this semigroup are $\binom{n}{k} \times S(n,k)$ matrices over S_k^0 having a single non-zero entry. Multiplication is defined by $w_1 \cdot w_2 := w_1 \Pi_{n,k} w_2$, where $\Pi_{n,k}$ is the $S(n,k) \times \binom{n}{k}$ sandwich matrix*

$$(\Pi_{n,k})_{\pi,P} = \begin{cases} \tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \le i \le k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Example 2.5.3.

$$\Pi_{4,3} = \begin{pmatrix} (id) & (id) & 0 & 0 \\ 0 & (id) & (id) & 0 \\ 0 & 0 & (id) & (id) \\ (id) & 0 & 0 & (123) \\ (id) & 0 & (23) & 0 \\ 0 & (id) & 0 & (12) \end{pmatrix},$$

where the ordering on the columns is $123, 124, 134, 234$, *and the ordering on the rows is* $1|2|34, 1|23|4, 12|3|4, 14|2|3, 1|24|3, 13|2|4$. *Now let* $w_1 = 2142$ *and* $w_2 = 4223$. *Then*

$$w_1 w_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (12) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (id) & (id) & 0 & 0 \\ 0 & (id) & (id) & 0 \\ 0 & 0 & (id) & (id) \\ (id) & 0 & 0 & (123) \\ (id) & 0 & (23) & 0 \\ 0 & (id) & 0 & (12) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (132) & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (12) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = 2114$$

Recall that a semigroup S with zero element 0 is called 0-*simple* if $S^2 \neq 0$ and 0 is the only proper two-sided ideal of S. A Rees matrix semigroup is *regular* if its sandwich matrix Π has no row or column of all zeros. The following result ([3], Theorem 3.5) classifies all finite 0-simple semigroups.

Theorem 2.5.4. *(Rees) A finite semigroup is* 0*-simple if and only if it is isomorphic with a regular Rees matrix semigroup over a group with zero.*

The analogous structure for algebras is called a *Munn matrix algebra.*

Definition 2.5.5. *Let A be an associative algebra, and Π an $n \times m$ matrix over A. The* Munn matrix algebra *$\mathcal{A} = \mathcal{M}(A; m, n; \Pi)$ is the set of all $m \times n$ matrices with entries in A, where multiplication is defined by $X \cdot Y := X\Pi Y$. Π is called the* sandwich matrix *for $\mathcal{A}$.*

Fact 2.5.6. *$A_{n,k}$ is isomorphic to the Munn matrix algebra $\mathcal{M}(\mathbb{C}S_k; \binom{[n]}{k}, S([n],k); \Pi)$. The elements of this semigroup are $\binom{n}{k} \times S(n,k)$ matrices over $\mathbb{C}S_k$. As with the corresponding Rees matrix semigroup, multiplication is defined by $w_1 \cdot w_2 := w_1 \Pi w_2$, where Π is the $S(n,k) \times \binom{n}{k}$ sandwich matrix*

$$(\Pi)_{\pi,P} = \begin{cases} \tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \leq i \leq k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.5.7. *(Clifford and Preston) ([3], Theorem 5.19) The Munn algebra $\mathcal{A} = \mathcal{M}(A; m, n; \Pi)$ over a finite dimensional associative algebra A is semisimple if and only if*

(i) A is semisimple, and

(ii) Π is non-singular, i.e., if and only if $n = m$ and Π is a unit in the ring of $m \times m$ matrices over A.

If this is a case, then $\mathcal{A} \cong (A)_m$, the algebra of $m \times m$ matrices with entries in A.

Example 2.5.8. *Take $n = 3$ and $k = 2$. Then*

$$\Pi_{3,2} = \begin{pmatrix} id & id & 0 \\ 0 & id & id \\ id & 0 & (12) \end{pmatrix},$$

where the columns are indexed by the subsets $\{1,2\}, \{1,3\}, \{2,3\}$ and the rows are indexed by the partitions $1|23, 12|3, 13|2$. All of the entries of $\Pi_{3,2}$ commute, so the determinant is well-defined. We have

$$\det \Pi = id((12) - 0) - id(0 - id) = id + (12) \neq 0.$$

Recall however that $\dim \sqrt{A_{3,2}} = 5$, in particular $A_{3,2}$ is not semisimple. Therefore it must be that Π does not have an inverse in the algebra of 3×3 matrices over $\mathbb{C}S_2$. Suppose that

$$\begin{pmatrix} id & id & 0 \\ 0 & id & id \\ id & 0 & (12) \end{pmatrix} \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = \begin{pmatrix} id & 0 & 0 \\ 0 & id & 0 \\ 0 & 0 & id \end{pmatrix}.$$

Then, in particular,

$$\begin{aligned} a + d &= id \\ d + g &= 0 \\ a + (12)g &= 0. \end{aligned}$$

Solving this system of equations comes down to solving

$$(id + (12))g = -id,$$

where $g \in \mathbb{C}S_3$. But it is easy to verify that this equation has no solution, and therefore $\Pi_{3,2}$ is singular.

Chapter 3

The x Deformation

Recall that the algebra $A_{n,k}$ is not semisimple for $k < n$. In this chapter we define the first of two deformations of $A_{n,k}$ aimed at making the algebra generically semisimple. We will require that the deformed algebra be associative, and have the same dimension as the original.

3.1 Definition

The Symmetric Group S_k acts on the maps of rank k by

$$\sigma w_{\pi,P,\phi} := w_{\pi,P,\sigma\phi}.$$

This is indeed an action, since $(\nu_1\nu_2)w_{\pi,P,\phi} = w_{\pi,P,(\nu_1\nu_2)\phi} = w_{\pi,P,\nu_1(\nu_2\phi)} = \nu_1(\nu_2 w_{\pi,P,\phi})$. We use this action to define a new multiplication in $A_{n,k}$.

Definition 3.1.1. *For $\lambda \vdash k$, let p_λ be the corresponding power-sum symmetric function in the variables $x_1, \ldots, x_k$. For $w_1, w_2 \in T_{n,k}$ define*

$$w_1 \circ w_2 := \sum_{\sigma \in S_k} p_{\rho(\sigma)} \sigma w_1 w_2,$$

where $\rho(\sigma)$ is the cycle type of σ. Let $A_{n,k}(\mathbf{x})$ be the complex algebra with basis $T_{n,k}$, where two elements multiply via $\circ$.

Example 3.1.2. *Let $w_1 = 1442$ and $w_2 = 3134$. Then $w_1 w_2 = 4142$, and*

$$\begin{aligned} w_1 \circ w_2 &= p_{1^3}4142 + p_{21}(1412 + 2124 + 4241) + p_3(1214 + 2421) \\ &= (x_1 + x_2 + x_3)^3 4142 + (x_1^2 + x_2^2 + x_3^2)(x_1 + x_2 + x_3)(1412 + 2124 + 4241) \\ &\quad + (x_1^3 + x_2^3 + x_3^3)(1214 + 2421) \end{aligned}$$

Note that if we choose values for the x_i so that $p_1 = 1$ and $p_i = 0$ for all $i \geq 2$, we recover the original multiplication in $A_{n,k}$. Such a specialization of the x_i must exist because the p_i are algebraically independent.

3.2 Associativity

Proposition 3.2.1. *The multiplication $\circ$ is associative.*

Proof. Let $w_1 = w_{\pi,S,\sigma}, w_2 = w_{\mu,T,\tau}$, and $w_3 = w_{\kappa,U,\psi} \in T_{n,k}$, and suppose there exist $\alpha, \beta \in S_k$ such that $t_i \in \pi_{\alpha(i)}, u_i \in \mu_{\beta(i)}$ for all $i = 1, \ldots, k$, i.e., α interpolates between w_2 and w_1 and β interpolates between w_3 and w_2. (If no such α and β exist then the product will be zero.) Then we have

$$\begin{aligned}
w_1 \circ (w_2 \circ w_3) &= w_{\pi,S,\sigma} \circ (w_{\mu,T,\tau} \circ w_{\kappa,U,\psi}) \\
&= w_{\pi,S,\sigma} \circ \big(\sum_{\gamma \in S_k} p_{\rho(\gamma)} \gamma w_{\mu,T,\tau} w_{\kappa,U,\psi} \big) \\
&= w_{\pi,S,\sigma} \circ \big(\sum_{\gamma \in S_k} p_{\rho(\gamma)} w_{\kappa,T,\gamma\tau\beta\psi} \big) \\
&= \sum_{\gamma \in S_k} p_{\rho(\gamma)} w_{\pi,S,\sigma} \circ w_{\kappa,T,\gamma\tau\beta\psi} \\
&= \sum_{\gamma \in S_k} p_{\rho(\gamma)} \sum_{\delta \in S_k} p_{\rho(\delta)} \delta w_{\pi,S,\sigma} w_{\kappa,T,\gamma\tau\beta\psi} \\
&= \sum_{\gamma,\delta \in S_k} p_{\rho(\gamma)} p_{\rho(\delta)} w_{\kappa,S,\delta\sigma\alpha\gamma\tau\beta\psi}.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
(w_1 \circ w_2) \circ w_3 &= (w_{\pi,S,\sigma} \circ w_{\mu,T,\tau}) \circ w_{\kappa,U,\psi} \\
&= \big(\sum_{\gamma \in S_k} p_{\rho(\gamma)} w_{\mu,S,\gamma\sigma\alpha\tau} \big) \circ w_{\kappa,U,\psi} \\
&= \sum_{\gamma,\delta \in S_k} p_{\rho(\gamma)} p_{\rho(\delta)} w_{\kappa,S,\delta\gamma\sigma\alpha\tau\beta\psi}.
\end{aligned}$$

To see that these two expressions are indeed the same, replace γ by $\sigma\alpha\gamma\alpha^{-1}\sigma^{-1}$. As γ ranges over all of S_k so does $\sigma\alpha\gamma\alpha^{-1}\sigma^{-1}$, and $\rho(\sigma\alpha\gamma\alpha^{-1}\sigma^{-1}) = \rho(\gamma)$ since conjugate permutations have the same cycle type. Thus we have

$$\begin{aligned}
\sum_{\gamma,\delta \in S_k} p_{\rho(\gamma)} p_{\rho(\delta)} w_{\kappa,S,\delta\gamma\sigma\alpha\tau\beta\psi} &= \sum_{\gamma,\delta \in S_k} p_{\rho(\sigma\alpha\gamma\alpha^{-1}\sigma^{-1})} p_{\rho(\delta)} w_{\kappa,S,\delta(\sigma\alpha\gamma\alpha^{-1}\sigma^{-1})\sigma\alpha\tau\beta\psi} \\
&= \sum_{\gamma,\delta \in S_k} p_{\rho(\gamma)} p_{\rho(\delta)} w_{\kappa,S,\delta\sigma\alpha\gamma\tau\beta\psi}.
\end{aligned}$$

□

3.3 Gram matrix

Let $M_{n,k}(\mathbf{x})$ be the Gram matrix for $A_{n,k}(\mathbf{x})$, defined by $(M_{n,k}(\mathbf{x}))_{u,v} = tr(u(\mathbf{x})v(\mathbf{x}))$, where for $w \in T_{n,k}$ we let $w(\mathbf{x})$ denote the matrix representing left $\circ$-multiplication by

w. Explicit calculations for small values of n and k suggest that $M_{n,k}(\mathbf{x})$ is still singular for $k < n$. Indeed it appears that $A_{n,k}(\mathbf{x})$ is no more semisimple than $A_{n,k}$, i.e., that even for generic values of the x_i we have $\dim\sqrt{A_{n,k}(\mathbf{x})} = \dim\sqrt{A_{n,k}}$. However, as we shall show in this section and the next, we find something interesting when we look at the eigenvalues of $M_{n,k}(\mathbf{x})$. We begin by computing the entries of the Gram matrix.

Proposition 3.3.1.

$$(M_{n,k}(\mathbf{x}))_{u,v} = \begin{cases} S(n,k)k! \displaystyle\sum_{\substack{\phi,\psi \in S_k \\ \phi\psi = \sigma}} p_{\rho(\phi)}p_{\rho(\psi)} & \sigma := \mathrm{ind}(uv) \in S_k, \ \textit{and} \\ \\ 0 & \textit{otherwise,} \end{cases}$$

where by ind(w) we mean the permutation of the k elements of Im(w) induced by w.

Note that the "otherwise" in this formula includes both the case that $uv = 0$ and the case that $uv \neq 0$ but does not induce a permutation on $\mathrm{Im}(uv)$.

Proof. First of all

$$\begin{aligned} (M_{n,k})_{uv} &= tr((u(\mathbf{x})v(\mathbf{x}))) \\ &= tr((\sum_{\sigma \in S_k} p_{\rho(\sigma)}\sigma u)(\sum_{\nu \in S_k} p_{\rho(\nu)}\nu v)) \\ &= \sum_{\sigma,\nu \in S_k} p_{\rho(\sigma)}p_{\rho(\nu)} tr(\sigma u \nu v). \end{aligned}$$

Now recall that

$$tr(\sigma u \nu v) = \begin{cases} S(n,k)k! & \sigma u \nu v \text{ has } k \text{ fixed points, and} \\ \\ 0 & \text{otherwise.} \end{cases}$$

□

As before, we shall normalize the Gram matrix by dividing by $S(n,k)k!$.

We claim that the entries of $M_{n,k}(\mathbf{x})$ can be expressed in terms of Schur symmetric functions in a particularly nice way. As evidence of this claim, consider the following example.

Example 3.3.2.

$$M_{3,2}(\mathbf{x}) = \begin{pmatrix}
0 & 0 & 0 & 0 & A & A & 0 & 0 & 0 & 0 & A & A & B & B & B & B & 0 & 0 \\
0 & A & 0 & 0 & 0 & A & 0 & 0 & 0 & A & 0 & A & B & 0 & B & 0 & B & B \\
0 & 0 & A & A & 0 & 0 & B & B & 0 & 0 & B & B & 0 & 0 & A & A & 0 & 0 \\
0 & 0 & A & A & A & A & B & B & 0 & 0 & 0 & 0 & B & B & 0 & 0 & 0 & 0 \\
A & 0 & 0 & A & 0 & 0 & B & 0 & B & B & 0 & B & 0 & 0 & A & 0 & 0 & A \\
A & A & 0 & A & 0 & A & B & 0 & B & 0 & 0 & 0 & B & 0 & 0 & 0 & B & 0 \\
0 & 0 & B & B & B & B & A & A & 0 & 0 & 0 & 0 & A & A & 0 & 0 & 0 & 0 \\
0 & 0 & B & B & 0 & 0 & A & A & 0 & 0 & A & A & 0 & 0 & B & B & 0 & 0 \\
0 & 0 & 0 & 0 & B & B & 0 & 0 & 0 & 0 & B & B & A & A & A & A & 0 & 0 \\
0 & A & 0 & 0 & B & 0 & 0 & 0 & 0 & A & B & 0 & 0 & A & 0 & A & B & B \\
A & 0 & B & 0 & 0 & 0 & 0 & A & B & B & A & 0 & 0 & 0 & 0 & B & 0 & A \\
A & A & B & 0 & B & 0 & 0 & A & B & 0 & 0 & 0 & 0 & A & 0 & 0 & B & 0 \\
B & B & 0 & B & 0 & B & A & 0 & A & 0 & 0 & 0 & A & 0 & 0 & 0 & A & 0 \\
B & 0 & 0 & B & 0 & 0 & A & 0 & A & A & 0 & A & 0 & 0 & B & 0 & 0 & B \\
B & B & A & 0 & A & 0 & 0 & B & A & 0 & 0 & 0 & 0 & B & 0 & 0 & A & 0 \\
B & 0 & A & 0 & 0 & 0 & 0 & B & A & A & B & 0 & 0 & 0 & 0 & A & 0 & B \\
0 & B & 0 & 0 & 0 & B & 0 & 0 & 0 & B & 0 & B & A & 0 & A & 0 & A & A \\
0 & B & 0 & 0 & A & 0 & 0 & 0 & 0 & B & A & 0 & 0 & B & 0 & B & A & A
\end{pmatrix},$$

where $A = p_1^4 + p_2^2 = 2(s_2^2 + s_{1^2}^2)$ *and* $B = 2p_1^2 p_2 = 2(s_2^2 - s_{1^2}^2)$.

Proposition 3.3.3.

$$(M_{n,k}(\mathbf{x}))_{i,j} = \begin{cases} \sum_{\lambda \vdash k} \frac{k!}{f^\lambda} \chi^\lambda(\mu) s_\lambda^2 & \text{if } w_i w_j \text{ induces a permutation of cycle type } \mu \text{ on } P(w_i), \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

The proof of this proposition requires that we first establish the following symmetric function identity.

Lemma 3.3.4. *Fix* $\sigma \in S_k$*, with* $\rho(\sigma) = \mu \vdash k$*. Then*

$$\sum_{\gamma \in S_k} p_{\rho(\gamma)}(\mathbf{x}) p_{\rho(\gamma^{-1}\sigma)}(\mathbf{y}) = \sum_{\lambda \vdash k} \frac{k!}{f^\lambda} \chi^\lambda(\mu) s_\lambda(\mathbf{x}) s_\lambda(\mathbf{y}). \tag{3.1}$$

To prove this identity we begin by proving the following sublemma.

Sublemma 3.3.5. *For* $\nu, \eta \vdash k$ *and* $\sigma \in S_k$ *with* $\rho(\sigma) = \mu$ *the number of* $\gamma \in S_k$ *such that* $\rho(\gamma) = \nu$ *and* $\rho(\gamma^{-1}\sigma) = \eta$ *is*

$$\frac{k!}{z_\nu z_\eta} \sum_{\lambda \vdash k} \frac{\chi^\lambda(\mu)\chi^\lambda(\nu)\chi^\lambda(\eta)}{f^\lambda},$$

where if $\lambda \vdash n$ *has* m_i *parts of size* i*,* $z_\lambda := \prod_{i \geq 1} i^{m_i} m_i!$.

Proof of sublemma: For $\lambda \vdash k$, define

$$\begin{aligned}
C_\lambda &:= \{g \in S_k | \rho(g) = \lambda\} \\
K_\lambda &:= \sum_{g \in C_\lambda} g \in \mathbb{C}S_k \\
F_\lambda &:= \frac{f^\lambda}{k!} \sum_{\mu \vdash k} \chi^\lambda(\mu) K_\mu \in \mathbb{C}S_k.
\end{aligned}$$

Note that $|C_\lambda| = \frac{k!}{z_\lambda}$.

Now we make use of two facts, the proofs of which are classical and will not be included here (see [13] or [8]).

Fact 3.3.6. *The F_λ are orthogonal idempotents in (the center of) $\mathbb{C}S_k$.*

Fact 3.3.7. *We can invert the defining equation for the F_λ to get*

$$K_\lambda = \frac{k!}{z_\lambda} \sum_{\mu \vdash k} \frac{\chi^\mu(\lambda)}{f^\mu} F_\mu.$$

Now, for $\nu, \eta \vdash k$ we have

$$\begin{aligned}
K_\nu K_\eta &= \left(\frac{k!}{z_\nu} \sum_{\lambda \vdash k} \frac{\chi^\lambda(\nu)}{f^\lambda} F_\lambda \right) \left(\frac{k!}{z_\eta} \sum_{w \vdash k} \frac{\chi^w(\eta)}{f^w} F_w \right) \quad \text{(by Fact 3.3.7)} \\
&= \frac{(k!)^2}{z_\nu z_\eta} \sum_{\lambda, w \vdash k} \frac{\chi^\lambda(\nu)\chi^w(\eta)}{f^\lambda f^w} F_\lambda F_w \\
&= \frac{(k!)^2}{z_\nu z_\eta} \sum_{\lambda \vdash k} \frac{\chi^\lambda(\nu)\chi^\lambda(\eta)}{(f^\lambda)^2} F_\lambda \quad \text{(by Fact 3.3.6)} \\
&= \frac{(k!)^2}{z_\nu z_\eta} \sum_{\lambda \vdash k} \frac{\chi^\lambda(\nu)\chi^\lambda(\eta)}{(f^\lambda)^2} \frac{f^\lambda}{k!} \sum_{\mu \vdash k} \chi^\lambda(\mu) K_\mu \\
&= \frac{k!}{z_\nu z_\eta} \sum_{\lambda, \mu \vdash k} \frac{\chi^\lambda(\nu)\chi^\lambda(\eta)\chi^\lambda(\mu)}{f^\lambda} K_\mu \\
&= \sum_{\mu \vdash k} \left(\frac{k!}{z_\nu z_\eta} \sum_{\lambda \vdash k} \frac{\chi^\lambda(\mu)\chi^\lambda(\nu)\chi^\lambda(\eta)}{f^\lambda} \right) K_\mu.
\end{aligned}$$

But the coefficient of K_μ in $K_\nu K_\eta$ is the number of $\gamma \in S_k$ such that $\rho(\gamma) = \nu$ and $\rho(\gamma^{-1}\sigma) = \eta$, where σ is some fixed element of C_μ. This proves the sublemma.

Proof. (of Lemma 3.3.4) Observe that the left-hand side of (3.1) is

$$\sum_{\gamma \in S_k} p_{\rho(\gamma)}(\mathbf{x}) p_{\rho(\gamma^{-1}\sigma)}(\mathbf{y}) = \sum_{\nu, \eta \vdash k} \begin{pmatrix} \#\gamma \in S_k \text{ s.t.} \\ \rho(\gamma) = \nu \text{ and} \\ \rho(\gamma^{-1}\sigma) = \eta \end{pmatrix} p_\nu(\mathbf{x}) p_\eta(\mathbf{y}), \tag{3.2}$$

while the right-hand side of (3.1) is

$$\begin{aligned}
\sum_{\lambda\vdash k}\frac{k!}{f^\lambda}\chi^\lambda(\mu)s_\lambda(\mathbf{x})s_\lambda(\mathbf{y}) &= \sum_{\lambda\vdash k}\frac{k!}{f^\lambda}\chi^\lambda(\mu)\left(\sum_{\nu\vdash k}\frac{\chi^\lambda(\nu)}{z_\nu}p_\nu(\mathbf{x})\right)\left(\sum_{\eta\vdash k}\frac{\chi^\lambda(\eta)}{z_\eta}p_\eta(\mathbf{x})\right)\\
&= \sum_{\lambda\vdash k}\frac{k!}{f^\lambda}\chi^\lambda(\mu)\sum_{\nu,\eta\vdash k}\frac{\chi^\lambda(\nu)\chi^\lambda(\eta)}{z_\nu z_\eta}p_\nu(\mathbf{x})p_\eta(\mathbf{y})\\
&= \sum_{\nu,\eta\vdash k}\left(\frac{k!}{z_\nu z_\eta}\sum_{\lambda\vdash k}\frac{\chi^\lambda(\mu)\chi^\lambda(\nu)\chi^\lambda(\eta)}{f^\lambda}\right)p_\nu(\mathbf{x})p_\eta(\mathbf{y}).
\end{aligned}$$

□

Propositions 3.3.1 and Lemma 3.3.4 together give Proposition 3.3.3.

3.4 Frobenius' Group Determinant

Definition 3.4.1. *Given a finite group G, let $\{x_g|g \in G\}$ be a set of commuting indeterminants, and let A be the $|G| \times |G|$ matrix defined by $(A)_{g,h} = x_{gh^{-1}}$. The* group determinant $\Theta(G)$ *is defined by $\Theta(G) = \det(A)$.*

Remark 3.4.2. *It was Dedekind who first defined the group determinant $\Theta(G)$. He showed that for abelian groups it factors completely into linear factors. He computed $\Theta(G)$ for a few small non-abelian groups, then passed the problem on to Frobenius. This was a key step in the development of representation theory, as it prompted Frobenius to invent a general character theory for arbitrary finite groups. Curtis [2] is an excellent source for the historical background of this theory.*

Proposition 3.4.3. *(Dedekind) Let G be an abelian group, and let $\hat{G}$ denote the set of irreducible characters of G. Then $\Theta(G)$ factors as*

$$\Theta(G) = \prod_{\chi\in\hat{G}}\left(\sum_{g\in G}\chi(g)x_g\right).$$

Proof. For a fixed $\chi \in \hat{G}$ multiply the g^{th} row of the matrix A by $\chi(g)$, then add up all the rows. In the h^{th} column we get

$$\sum_{g\in G}\chi(g)x_{gh^{-1}} = \left(\sum_{g'\in G}\chi(g')x_{g'}\right)\chi(h).$$

Thus $\Theta(G)$ is divisible by $\sum_{g\in G}\chi(g)x_g$ for each $g \in G$. Comparing powers of the x_g shows that this exhausts all the factors of $\Theta(G)$, and the constant term is clearly 1. □

Theorem 3.4.4. *(Frobenius) Let G be an arbitrary finite group. Then $\Theta(G)$ is a polynomial with integer coefficients in the indeterminants x_a, and factors over $\mathbb{C}$ as the product of irreducible polynomials Φ, each occurring with some multiplicity f:*

$$\Theta = \prod (\Phi)^f.$$

The number of irreducible factors Φ is the number of conjugacy classes in G. The degree of each factor Φ is equal to the multiplicity with which it occurs in the factorization of Θ:

$$deg(\Phi) = f.$$

If one sets $x_a = x_b$ whenever a and b are conjugate in G, then each factor of the specialized polynomial Θ can be expressed in the form

$$\Phi = (\xi)^f,$$

where $\xi = \frac{1}{f}\sum_i h_i \chi(i) x_i$ where $h_i = |C_i|$ and χ is the irreducible character corresponding to ξ.

We apply Frobenius' result to our Gram matrix to obtain the following factorization.

Proposition 3.4.5.

$$\det\left(\frac{1}{n!}M_{n,n}(\mathbf{x})\right) = \pm \prod_{\lambda \vdash n} \left(\left(\frac{n!}{f^\lambda} s_\lambda \right)^2 \right)^{(f^\lambda)^2},$$

where the sign is given by $(-1)^{\frac{1}{2}\left(n! - \sum_\lambda f^\lambda\right)}$.

Proof. By Proposition 3.3.7 the (u,v)-entry of $M_{n,n}(\mathbf{x})$ only depends on the conjugacy class of uv. If μ is the partition indexing this conjugacy class, let x_μ denote the corresponding entry of the matrix. Our determinant is then a specialization of Θ, up to a sign that comes from switching column w with column w^{-1} for each pair $\{w, w^{-1}\}$. By

Theorem 3.4.1 we have

$$\begin{aligned}
\det\left(\frac{1}{n!}M_{n,n}(\mathbf{x})\right) &= \pm\prod_{\lambda\vdash n}(\xi_\lambda)^{(f^\lambda)^2}\\
&= \pm\prod_{\lambda\vdash n}\left(\frac{1}{f^\lambda}\sum_{\mu\vdash n}\frac{n!}{z_\mu}\chi^\lambda(\mu)x_\mu\right)^{(f^\lambda)^2}\\
&= \pm\prod_{\lambda\vdash n}\left(\frac{1}{f^\lambda}\sum_{\mu\vdash n}\frac{n!}{z_\mu}\chi^\lambda(\mu)\sum_{\nu\vdash n}\frac{n!}{f^\nu}\chi^\nu(\mu)s_\nu^2\right)^{(f^\lambda)^2}\\
&= \pm\prod_{\lambda\vdash n}\left(\frac{1}{f^\lambda}\sum_{\nu\vdash n}\frac{n!}{f_\nu}\left(\sum_{\mu\vdash n}\frac{n!}{f^\mu}\chi^\lambda(\mu)\chi^\nu(\mu)s_\nu^2\right)\right)^{(f^\lambda)^2}\\
&= \pm\prod_{\lambda\vdash n}\left(\frac{1}{f^\lambda}\sum_{\nu\vdash n}\frac{n!}{f^\nu}n!\delta_{\lambda,\nu}s_\nu^2\right)^{(f^\lambda)^2} \quad \text{(by row orthogonality)}\\
&= \pm\prod_{\lambda\vdash n}\left((\frac{n!}{f^\lambda}s_\lambda)^2\right)^{(f^\lambda)^2}.
\end{aligned}$$

The sign is given by

$$(-1)^{\#\text{ pairs } \{w,w^{-1}\}} = (-1)^{\frac{1}{2}(\#\text{ non-involutions in } S_n)} = (-1)^{\frac{1}{2}(n!-\sum_\lambda f^\lambda)}.$$

□

Corollary 3.4.6. *The algebra $A_{n,k}(\mathbf{x})$ is semisimple if and only if $\mathbf{x}$ is chosen so that $s_\lambda = \sum_{\mu\vdash n}\frac{1}{z_\mu}\chi^\lambda(\mu)p_\mu \neq 0$ for all $\lambda \vdash n$. Since this condition is polynomial in the indeterminants x_i, $A_{n,k}(\mathbf{x})$ is semisimple for generic $\mathbf{x}$.*

3.5 Schur function eigenvalues

Proposition 3.4.2 suggests the following stronger result.

Theorem 3.5.1. *Let $M_{n,k}(\mathbf{x})$ be the normalized Gram matrix for $A_{n,k}(\mathbf{x})$. Then for $k = n$ we have the following.*

e-value of $M_{n,n}(\mathbf{x})$	*multiplicity*
$\left(\frac{n!}{f^\lambda}s_\lambda\right)^2$	$f^\lambda + \binom{f^\lambda}{2}$
$-\left(\frac{n!}{f^\lambda}s_\lambda\right)^2$	$\binom{f^\lambda}{2}$

Since $f^\lambda + 2\binom{f^\lambda}{2} = (f^\lambda)^2$ and $\sum_{\lambda\vdash n}(f^\lambda)^2 = n!$ this statement characterizes all the eigenvalues of $M_{n,n}(\mathbf{x})$.

Proof. Define $C_{n,n}(\mathbf{x})$ by

$$(C_{n,n}(\mathbf{x}))_{w_i,w_j} := \sum_{\lambda \vdash n} \frac{n!}{f^\lambda} \chi^\lambda(\mu) s_\lambda^2,$$

where $\mu = \rho(w_i w_j^{-1})$. Note that $C_{n,n}(\mathbf{x}) = M_{n,n}(\mathbf{x})E$, where E is the involution flipping columns w and w^{-1} for each $w \in S_n$.

Let $a_{i,j}^\lambda : S_n \longrightarrow \mathbb{C}, \lambda \vdash n, 1 \leq i,j \leq f^\lambda$, be the matrix entry functions for the irreducible unitary matrix representations of S_n. (It is a fact that S_n can be represented by unitary matrices.)

For each $\lambda \vdash n$ and $1 \leq i,j \leq f^\lambda$ define the vector $\mathbf{v}_{i,j}^\lambda$ by $(\mathbf{v}_{i,j}^\lambda)_g = a_{i,j}^\lambda(g)$.

Now we need the following lemma, true for an arbitrary group G, on the orthogonality of the matrix entries of the irreducible representations of G.

Lemma 3.5.2. *(See [13], Section 1.9.) Let $\lambda, \mu \vdash n, a_{i,j}^\lambda, a_{i,j}^\mu$ as above, and $h \in S_n$. Then*

$$\sum_{g \in S_n} a_{i,k}^\lambda(hg) a_{l,j}^\mu(g^{-1}) = \delta_{\lambda,\mu}\delta_{k,l} \frac{n!}{f^\lambda} a_{i,j}^\lambda(h).$$

To prove the theorem, let $C_{n,n}$ and $\mathbf{v}_{i,j}^\mu$ be as above.

$$\begin{aligned}
(C_{n,n}\mathbf{v}_{i,j}^\mu)_u &= \sum_{w \in S_n} (C_{n,n})_{u,w} (\mathbf{v}_{i,j}^\mu)_w \\
&= \sum_{w \in S_n} \left(\sum_{\lambda \vdash n} \frac{n!}{f^\lambda} \chi^\lambda(\rho(uw^{-1})) s_\lambda^2 \right) a_{i,j}^\mu(w) \\
&= \sum_{w \in S_n} \sum_{\lambda \vdash n} \frac{n!}{f^\lambda} \sum_k a_{k,k}^\lambda(uw^{-1}) a_{i,j}^\mu(w) s_\lambda^2 \\
&= \sum_{\lambda \vdash n} \frac{n!}{f^\lambda} \sum_k \left(\sum_{w \in S_n} a_{k,k}^\lambda(uw^{-1}) a_{i,j}^\mu(w) \right) s_\lambda^2 \\
&= \sum_{\lambda \vdash n} \frac{n!}{f^\lambda} \sum_k \left(\delta_{\lambda,\mu}\delta_{k,i} \frac{n!}{f^\mu} a_{i,j}^\mu(u) \right) s_\lambda^2 \qquad \text{(by Lemma 3.5.2)} \\
&= \frac{n!}{f_\mu} \frac{n!}{f_\mu} a_{i,j}^\mu(u) s_\mu^2 \\
&= \left(\frac{n!}{f^\mu} s_\mu \right)^2 a_{i,j}^\mu(u).
\end{aligned}$$

So $\mathbf{v}_{i,j}^\mu$ is an eigenvector for $C_{n,n}$ with eigenvalue $\left(\frac{n!}{f^\mu} s_\mu\right)^2$. Now recall that $C_{n,n}(\mathbf{x}) = M_{n,n}(\mathbf{x})E$, where E is the involution flipping columns w and w^{-1} for each $w \in S_n$. Note that $M_{n,n}(\mathbf{x})$ and $C_{n,n}(\mathbf{x})$ are both symmetric since $\rho(w_2 w_1) = \rho(w_2^{-1} w_2 w_1 w_2) = \rho(w_1 w_2)$ (conjugation preserves cycle structure) and $\rho(w_2 w_1^{-1}) = \rho((w_1 w_2^{-1})^{-1}) = \rho(w_1 w_2^{-1})$ (taking inverses preserves cycle structure), and E is obviously symmetric.

So $C_{n,n}E = M_{n,n} = (M_{n,n})^t = (C_{n,n}E)^t = E^t(C_{n,n})^t = EC_{n,n}$.

Since $C_{n,n}$ and E commute they are simultaneously diagonalizable, so a vector that is both an eigenvector for $C_{n,n}$ and an eigenvector for E will be an eigenvector for $C_{n,n}E = M_{n,n}$

Now define the following set of vectors.

$$\begin{aligned}
\mathbf{u}^\lambda_{i,i} &:= \mathbf{v}^\lambda_{i,i}, 1 \le i \le f^\lambda \\
\mathbf{u}^\lambda_{i,j} &:= \mathbf{v}^\lambda_{i,j} + \mathbf{v}^\lambda_{j,i}, 1 \le i < j \le f^\lambda \\
\mathbf{w}^\lambda_{i,j} &:= \mathbf{v}^\lambda_{i,j} - \mathbf{v}^\lambda_{j,i}, 1 \le i < j \le f^\lambda
\end{aligned}$$

Each vector is clearly an eigenvector for $C_{n,n}$ (with eigenvalue $\left(\frac{n!}{f^\lambda}s_\lambda\right)^2$). But what about for E?

The key is that we are working with *unitary* representations. Moreover, the matrix entries are real, so the representing matrices satisfy $A^{-1} = A^t$. Thus $a^\lambda_{i,j}(g^{-1}) = a^\lambda_{j,i}(g)$. So $(E\mathbf{v}^\lambda_{i,j})_g = (\mathbf{v}^\lambda_{i,j})_{g^{-1}} = a^\lambda_{i,j}(g^{-1}) = a^\lambda_{j,i}(g) = (\mathbf{v}^\lambda_{j,i})_g$, i.e., $E\mathbf{v}^\lambda_{i,j} = \mathbf{v}^\lambda_{j,i}$.

But then $E\mathbf{u}^\lambda_{i,i} = \mathbf{u}^\lambda_{i,i}$, $E\mathbf{u}^\lambda_{i,j} = \mathbf{u}^\lambda_{i,j}$, and $E\mathbf{w}^\lambda_{i,j} = -\mathbf{w}^\lambda_{i,j}$.

So

$$\begin{aligned}
M_{n,n}\mathbf{u}^\lambda_{i,i} = C_{n,n}E\mathbf{u}^\lambda_{i,i} &= C_{n,n}\mathbf{u}^\lambda_{i,i} = \left(\frac{n!}{f^\lambda}s_\lambda\right)^2 \mathbf{u}^\lambda_{i,i} \\
M_{n,n}\mathbf{u}^\lambda_{i,j} = C_{n,n}E\mathbf{u}^\lambda_{i,j} &= C_{n,n}\mathbf{u}^\lambda_{i,j} = \left(\frac{n!}{f^\lambda}s_\lambda\right)^2 \mathbf{u}^\lambda_{i,j} \\
M_{n,n}\mathbf{w}^\lambda_{i,j} = C_{n,n}E\mathbf{w}^\lambda_{i,j} &= C_{n,n}(-\mathbf{w}^\lambda_{i,j}) = -\left(\frac{n!}{f^\lambda}s_\lambda\right)^2 \mathbf{w}^\lambda_{i,j}.
\end{aligned}$$

There are f^λ of the $\mathbf{u}^\lambda_{i,i}$, and $\binom{f^\lambda}{2}$ each of the $\mathbf{u}^\lambda_{i,j}$ and $\mathbf{w}^\lambda_{i,j}$. $\sum\limits_{\lambda \vdash n} f^\lambda + 2\binom{f^\lambda}{2} = n!$, and the vectors are clearly linearly independent since the $\mathbf{v}^\lambda_{i,j}$ are orthogonal. So we have found a full set of eigenvectors for $M_{n,n}$ and the proposition then follows. □

Note that we have $\mathbf{v}^\lambda_{i,j} \cdot \mathbf{v}^\mu_{k,l} = 0$ for $\lambda \ne \mu$ by taking $h = g^{-1}$ in the lemma, but within the particular eigenspaces the vectors are not orthogonal.

Conjecture 3.5.3. *The non-zero eigenvalues of $M_{n,k}(\mathbf{x})$, $k < n$, are constant multiples of the squares of the Schur functions s_λ for $\lambda \vdash k$. The multiplicity of s^2_λ is $(\binom{n}{k}f^\lambda)^2$ for $\lambda \vdash k$, $\lambda \ne 1^k$, and $\binom{n-1}{k-1}^2$ for $\lambda = 1^k$.*

(The conjectured multiplicities come from Putcha's results; they are the dimensions of the irreducible characters for $A_{n,k}$.)

This conjecture is difficult to check even by computer for $n > 4$. The following table gives some sample data.

n	k	eigenvalues (listed by multiplicity)
2	2	$(2s_{1^2}^2)^1, (4s_2^2)^1$
2	1	$0^1, (2s_1^2)^1$
3	3	$(-9s_{21}^2)^1, (36s_{1^3}^2)^1, (9s_{21}^2)^3, (36s_3^2)^1$
3	2	$(-12s_{1^2}^2)^1, (-8s_2^2)^2, (-4s_2^2)^1, 0^5, (4s_2^2)^3, (8s_2^2)^2, (12s_{1^2}^2)^3, (16s_2^2)^1$
3	1	$0^2, (3s_1^2)^1$
4	4	$(-64s_{31}^2)^3, (-144s_{2^2}^2)^1, (-64s_{21^2}^2)^3, (576s_{1^4}^2)^1, (64s_{21^2}^2)^6,$ $(144s_{2^2}^2)^3, (64s_{31}^2)^6, (576s_4^2)^1$
4	3	(computations too big)
4	2	$(-32s_{1^2}^2)^3, (-\sqrt{448}s_2^2)^5, (-8s_2^2)^{10}, 0^{39}, (8s_2^2)^{15}, (\sqrt{448}s_2^2)^5, (32s_{1^2}^2)^6, (56s_2^2)^1$
4	1	$0^3, (4s_1^2)^1$

One possible difficulty in approaching this conjecture is that we do not have inverses in $T_{n,k}$ for $k < n$, and thus cannot interpret the determinant of the Gram matrix as some sort of "semigroup determinant" in order to use some (unknown) Frobenius-type theorem for semigroups.

3.6 x deformation of the Rook Monoid algebra $\mathbb{C}R_{n,k}$

In this section we define an analogous **x**-multiplication on the elements of $R_{n,k}$.

The Symmetric Group S_k acts on $R_{n,k}$ by

$$\gamma w_{S,T,\sigma} := w_{S,T,\gamma\sigma}.$$

Now, as we did with the Full Transformation Semigroup Algebra we define

$$w_1 \circ w_2 := \sum_{\gamma \in S_k} p_{\rho(\gamma)} \gamma w_1 w_2.$$

Proposition 3.6.1. *The multiplication $\circ$ is associative.*

Proof. Clearly we need only check the case in which the domains and ranges of the various maps line up so that the product is non-zero. Let $w_1 = w_{R,S,\sigma_1}, w_2 = w_{Q,R,\sigma_2}$, and $w_3 = w_{P,Q,\sigma_3}$. Then we have

$$\begin{aligned} w_1 \circ (w_2 \circ w_3) &= w_{R,S,\sigma_1} \circ (w_{Q,R,\sigma_2} \circ w_{P,Q,\sigma_3}) \\ &= w_{R,S,\sigma_1} \circ \left(\sum_{\gamma \in S_k} p_{\rho(\gamma)} \gamma w_{Q,R,\sigma_2} w_{P,Q,\sigma_3} \right) \\ &= w_{R,S,\sigma_1} \circ \left(\sum_{\gamma \in S_k} p_{\rho(\gamma)} w_{P,R,\gamma\sigma_2\sigma_3} \right) \\ &= \sum_{\gamma \in S_k} p_{\rho(\gamma)} \sum_{\nu \in S_k} p_{\rho(\nu)} w_{P,S,\nu\sigma_1\gamma\sigma_2\sigma_3} \\ &= \sum_{\gamma,\nu \in S_k} p_{\rho(\gamma)} p_{\rho(\nu)} w_{P,S,\nu\sigma_1\gamma\sigma_2\sigma_3} \end{aligned}$$

On the other hand,

$$\begin{aligned} (w_1 \circ w_2) \circ w_3 &= (w_{R,S,\sigma_1} \circ w_{Q,R,\sigma_2}) \circ w_{P,Q,\sigma_3} \\ &= \left(\sum_{\gamma \in S_k} p_{\rho(\gamma)} w_{Q,S,\gamma\sigma_1\sigma_2} \right) \circ w_{P.Q.\sigma_3} \\ &= \sum_{\gamma \in S_k} p_{\rho(\gamma)} \sum_{\nu \in S_k} p_{\rho(\nu)} w_{P,S,\nu\gamma\sigma_1\sigma_2\sigma_3} \\ &= \sum_{\gamma,\nu \in S_k} p_{\rho(\gamma)} p_{\rho(\nu)} w_{P,S,\nu\gamma\sigma_1\sigma_2\sigma_3} \\ &= \sum_{\gamma,\nu \in S_k} p_{\rho(\gamma)} p_{\rho(\nu)} w_{P,S,\nu\sigma_1\gamma\sigma_2\sigma_3} \\ &= w_{R,S,\sigma_1} \circ (w_{Q,R,\sigma_2} \circ w_{P,Q,\sigma_3}), \end{aligned}$$

where in the second to last step we replace γ by $\sigma_1\gamma\sigma_1^{-1}$, which does not change the cycle type. □

Let $\mathbb{C}R_{n,k}(\mathbf{x})$ be the associative algebra with this new $\circ$ multiplication, and consider the Gram matrix $M_{n,k}(\mathbf{x})$.

For $w \in R_{n,k}$, let $w(\mathbf{x})$ be the matrix representing left $\mathbf{x}$-multiplication by w.

Since $w_1 \circ w_2 = \sum_{\gamma \in S_k} p_{\rho(\gamma)} \gamma w_1 w_2$, we have

$$w(\mathbf{x}) = \sum_{\gamma \in S_k} p_{\rho(\gamma)} \gamma w.$$

So

$$\begin{aligned}
\operatorname{tr}\left(w_1(\mathbf{x})w_2(\mathbf{x})\right) &= \operatorname{tr}\left(\left(\sum_{\gamma\in S_k} p_{\rho(\gamma)}\gamma w_1\right)\left(\sum_{\nu\in S_k} p_{\rho(\nu)}\nu w_2\right)\right)\\
&= \operatorname{tr}\left(\sum_{\gamma,\nu\in S_k} p_{\rho(\gamma)}p_{\rho(\nu)}\gamma w_1\nu w_2\right)\\
&= \sum_{\gamma,\nu\in S_k} p_{\rho(\gamma)}p_{\rho(\nu)}\operatorname{tr}\left(\gamma w_1\nu w_2\right).
\end{aligned}$$

The only partial transformations of $[n]$ with non-zero trace are the "partial identities", i.e., maps of the form $w_{S,S,\mathrm{id}}$. Moreover

$$\begin{aligned}
\operatorname{tr}(w_{S,S,\mathrm{id}}) &= \# w_{A,B,\phi} \text{ such that } w_{S,S,\mathrm{id}}w_{A,B,\phi} = w_{A,B,\phi}\\
&= \binom{n}{k}k!,
\end{aligned}$$

since we must have $B = S$, while A and ϕ are arbitrary.

So $\operatorname{tr}\left(w_1(\mathbf{x})w_2(\mathbf{x})\right) = 0$ unless $\mathrm{Im}(w_2) = \mathrm{Dom}(w_1)$ and $\mathrm{Im}(w_1) = \mathrm{Dom}(w_2)$.

In that special case we have

$$\begin{aligned}
\operatorname{tr}\left(w_{T,S,\sigma}(\mathbf{x})w_{S,T,\sigma}(\mathbf{x})\right) &= \binom{n}{k}k! \sum_{\substack{\gamma,\nu\in S_k\\ \gamma\sigma\nu\phi=\mathrm{id}}} p_{\rho(\gamma)}p_{\rho(\nu)}\\
&= \binom{n}{k}k! \sum_{\substack{\gamma,\nu\in S_k\\ \gamma\nu\sigma\phi=\mathrm{id}}} p_{\rho(\gamma)}p_{\rho(\nu)} \text{ (replace } \nu \text{ by } \sigma^{-1}\nu\sigma)\\
&= \binom{n}{k}k! \sum_{\substack{\gamma,\nu\in S_k\\ \gamma\nu=\sigma\phi}} p_{\rho(\gamma)}p_{\rho(\nu)} \text{ (replace } \gamma \text{ by } \nu^{-1}, \nu \text{ by } \gamma^{-1})\\
&= \binom{n}{k}k! \sum_{\lambda\vdash k} \frac{k!}{f^\lambda}\chi^\lambda\left(\rho(\sigma\phi)\right)s_\lambda^2,
\end{aligned}$$

by Proposition 3.3.3.

Thus we have

$$\left(M_{n,k}(\mathbf{x})\right)_{w_{S,T,\sigma},w_{A,B,\phi}} = \begin{cases} \sum\lambda\vdash k\frac{k!}{f^\lambda}\chi^\lambda(\mu)s_\lambda^2 & \text{if } B=S, A=T, \text{ and } \mu:=\rho(\sigma\phi)\\ 0 & \text{otherwise.}\end{cases}$$

For example,

$$M_{3,2}(\mathbf{x}) = 6 \begin{pmatrix}
A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
B & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & B & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & A & 0 & 0 & 0 & 0 \\
0 & 0 & A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & B & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & A & 0 & 0 \\
0 & 0 & 0 & 0 & A & B & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & B & A & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & A & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & A & B \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & B & A
\end{pmatrix},$$

where $A = p_1^4 + p_2^2 = 2(s_2^2 + s_{1^2}^2)$ and $B = 2p_1^2 p_2 = 2(s_2^2 - s_{1^2}^2)$.

Notice that if we order the elements of $R_{n,k}$ so that the maps with domain S and range T are paired with the maps with domain T and range S we get a block matrix of the form

$$\binom{n}{k} k! \begin{pmatrix}
M & & & & & & & & & & & \\
& M & & & & & & & & & & \\
& & \ddots & & & & & & & & & \\
& & & M & & & & & & & & \\
& & & & 0 & M & & & & & & \\
& & & & M & 0 & & & & & & \\
& & & & & & 0 & M & & & & \\
& & & & & & M & 0 & & & & \\
& & & & & & & & \ddots & & & \\
& & & & & & & & & & 0 & M \\
& & & & & & & & & & M & 0
\end{pmatrix},$$

where $M = M_{k,k}(\mathbf{x})$ is the scaled Gram matrix for the group algebra $\mathbb{C}S_k(\mathbf{x})$.

Using Theorem 3.5.1 and the work above we get

Theorem 3.6.2. *Let $\mathbf{v}_{i,j}^{\lambda}$ be as above, where $\lambda \vdash k, 1 \leq i, j \leq f^{\lambda}$. For $S \in \binom{[n]}{k}$ define $\mathbf{w}_{i,j}^{\lambda,S}$ by*

$$\left(\mathbf{w}_{i,j}^{\lambda,S}\right)_{A,B,\phi} := \begin{cases} (\mathbf{v}_{i,j}^{\lambda})_{\phi} := a_{i,j}^{\lambda}(\phi) & \textit{if } A = B = S, \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

For $S,T \in \binom{[n]}{k}$, $S \neq T$, *define* $\mathbf{w}^{\lambda,S,T}_{i,j,+}$ *by*

$$\left(\mathbf{w}^{\lambda,S,T}_{i,j,+}\right)_{A,B,\phi} := \begin{cases} (\mathbf{v}^{\lambda}_{i,j})_{\phi} := a^{\lambda}_{i,j}(\phi) & \text{if } A = S \text{ and } B = T \\ & \text{or if } A = T \text{ and } B = S, \text{ and} \\ 0 & \text{otherwise,} \end{cases}$$

and define $\mathbf{w}^{\lambda,S,T}_{i,j,-}$ *by*

$$\left(\mathbf{w}^{\lambda,S,T}_{i,j,-}\right)_{A,B,\phi} := \begin{cases} (\mathbf{v}^{\lambda}_{i,j})_{\phi} := a^{\lambda}_{i,j}(\phi) & \text{if } A = S \text{ and } B = T, \\ -(\mathbf{v}^{\lambda}_{i,j})_{\phi} := -a^{\lambda}_{i,j}(\phi) & \text{if } A = T \text{ and } B = S, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathbf{w}^{\lambda,S}_{i,j}$ *and* $\mathbf{w}^{\lambda,S,T}_{i,j,+}$ *are eigenvectors for the scaled Gram matrix* $M_{n,k}(\mathbf{x})$ *with eigenvalue* $\left(\frac{k!}{f^{\lambda}}s_{\lambda}\right)^2$, *and* $\mathbf{w}^{\lambda,S,T}_{i,j,-}$ *is an eigenvector for* $M_{n,k}(\mathbf{x})$ *with eigenvalue* $-\left(\frac{k!}{f^{\lambda}}s_{\lambda}\right)^2$. *Together, they make up a complete set of eigenvectors for* $M_{n,k}(\mathbf{x})$.

Proof. First we need the following Lemma, which is simple to verify.

Lemma 3.6.3. *Let* A *be a matrix with eigenvector* $\mathbf{v}$ *and corresponding eigenvalue* λ. *Let* B *be the block matrix* $\begin{pmatrix} 0 & A \\ A & 0 \end{pmatrix}$. *Define* $\mathbf{w}_+$ *and* $\mathbf{w}_-$ *by concatenation as follows.*

$$\mathbf{w}_+ := \begin{pmatrix} \mathbf{v} \\ \mathbf{v} \end{pmatrix} \text{ and } \mathbf{w}_- := \begin{pmatrix} \mathbf{v} \\ -\mathbf{v} \end{pmatrix}.$$

Then $\mathbf{w}_+$ *and* $\mathbf{w}_-$ *are eigenvectors for* B *with eigenvectors* λ *and* $-\lambda$, *respectively.*

In our case we have

$$M_{n,k}(\mathbf{x}) = \binom{n}{k} k! \begin{pmatrix} M & & & & & & & & & & & \\ & M & & & & & & & & & & \\ & & \ddots & & & & & & & & & \\ & & & M & & & & & & & & \\ & & & & 0 & M & & & & & & \\ & & & & M & 0 & & & & & & \\ & & & & & & 0 & M & & & & \\ & & & & & & M & 0 & & & & \\ & & & & & & & & \ddots & & & \\ & & & & & & & & & 0 & M \\ & & & & & & & & & M & 0 \end{pmatrix},$$

where $M = M_{k,k}(\mathbf{x})$.

The blocks are indexed by ordered pairs of subsets (S,T), and we have listed them first where $S = T$, and then in pairs $(S,T),(T,S)$.

So

$$
\begin{aligned}
M_{n,k}(\mathbf{x})\mathbf{w}_{i,j}^{\lambda,S} &= \left(\begin{array}{ccc|cccc}
\ddots & & & & & & \\
& M & & & & & \\
& & \ddots & & & & \\
\hline
& & & \ddots & & & \\
& & & & 0 & M & \\
& & & & M & 0 & \\
& & & & & & \ddots
\end{array}\right)
\left(\begin{array}{c}
0 \\ \vdots \\ \mathbf{v}_{i,j}^{\lambda} \\ \vdots \\ 0 \\
\hline
0 \\ \vdots \\ 0
\end{array}\right) \\
&= \left(\begin{array}{c}
0 \\ \vdots \\ M\mathbf{v}_{i,j}^{\lambda} \\ \vdots \\ 0 \\
\hline
0 \\ \vdots \\ 0
\end{array}\right) \\
&= \left(\frac{k!}{f^{\lambda}} s_{\lambda}\right)^2 \mathbf{w}_{i,j}^{\lambda,S},
\end{aligned}
$$

while

$$
\begin{aligned}
M_{n,k}(\mathbf{x})\mathbf{w}_{i,j,+}^{\lambda,S,T} &= \left(\begin{array}{c|c}
\ddots & \\
\quad M & \\
\quad\quad \ddots & \\
\hline
 & \ddots \\
 & \begin{matrix} 0 & M \\ M & 0 \end{matrix} \\
 & \quad\quad \ddots
\end{array}\right)
\left(\begin{array}{c}
0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \mathbf{v}_{i,j}^{\lambda} \\ \mathbf{v}_{i,j}^{\lambda} \\ 0 \\ \vdots \\ 0
\end{array}\right) \\
&= \left(\begin{array}{c}
0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ M\mathbf{v}_{i,j}^{\lambda} \\ M\mathbf{v}_{i,j}^{\lambda} \\ 0 \\ \vdots \\ 0
\end{array}\right) \\
&= \left(\frac{k!}{f^{\lambda}} s_{\lambda}\right)^2 \mathbf{w}_{i,j}^{\lambda,S},
\end{aligned}
$$

and

$$M_{n,k}(\mathbf{x})\mathbf{w}^{\lambda,S,T}_{i,j,-} = \left(\begin{array}{ccc|ccc} \ddots & & & & & \\ & M & & & & \\ & & \ddots & & & \\ \hline & & & \ddots & & \\ & & & & \begin{matrix} 0 & M \\ M & 0 \end{matrix} & \\ & & & & & \ddots \end{array}\right) \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ \mathbf{v}^\lambda_{i,j} \\ -\mathbf{v}^\lambda_{i,j} \\ 0 \\ \vdots \\ 0 \end{array}\right)$$

$$= \left(\begin{array}{c} 0 \\ \vdots \\ 0 \\ \hline 0 \\ \vdots \\ 0 \\ -M\mathbf{v}^\lambda_{i,j} \\ M\mathbf{v}^\lambda_{i,j} \\ 0 \\ \vdots \\ 0 \end{array}\right)$$

$$= -\left(\frac{k!}{f^\lambda}s_\lambda\right)^2 \mathbf{w}^{\lambda,S}_{i,j}.$$

So $\mathbf{w}^{\lambda,S}_{i,j}$, $\mathbf{w}^{\lambda,S,T}_{i,j,+}$, and $\mathbf{w}^{\lambda,S,T}_{i,j,-}$ are eigenvectors as claimed, and since

$$\sum_{\lambda\vdash k}\left(\binom{n}{k}(f^\lambda)^2 + 2\frac{1}{2}\left(\binom{n}{k}^2 - \binom{n}{k}\right)(f^\lambda)^2\right) = \sum_{\lambda\vdash k}\binom{n}{k}^2(f^\lambda)^2 = \binom{n}{k}^2 k! = |R_{n,k}|,$$

and they are obviously linearly independent, they form a complete set of eigenvectors for $M_{n,k}(\mathbf{x})$. □

We can define a similar multiplication for any group algebra.

Definition 3.6.4. *Let G be a finite group and $f : G \longrightarrow S_n$ a (faithful) permutation representation of G. (We may take n as small as possible by stipulating that no element of $[n]$ is fixed by all the $f(g)$.) In the group algebra $\mathbb{C}G$ define*

$$g_1 \circ g_2 = \sum_{h\in G} p_{\rho(h)} h g_1 g_2,$$

where $\rho(h) = \rho(f(h))$ *is the cycle type of* $f(h) \in S_n$.

This multiplication is associative; indeed the proof is identical to the proof for the Rook Monoid algebra $\mathbb{C}R_{n,k}$. Defining the Gram matrix $M(\mathbf{x})$ as usual by $(M(\mathbf{x}))_{g_1,g_2} = \operatorname{tr}(g_1 \circ g_2) = \operatorname{tr}(g_1(\mathbf{x})g_2(\mathbf{x}))$, we have

Proposition 3.6.5. $(M(\mathbf{x}))_{g_1,g_2} = |G| \sum_{h \in G} p_{\rho(h)} p_{\rho(h^{-1}g_1g_2)}$.

Computations for small examples appear to show that we do not have an analogue of Theorem 3.5.1 for general groups, most likely because we do not have a nice analogue of the symmetric function identity of Lemma 3.3.4. Schur functions are intimately connected to the representation theory of the Symmetric Group and it may be too much to expect that they appear in the Gram matrix for other groups.

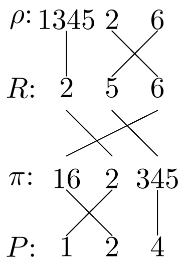
Chapter 4

The x Deformation

4.1 Definition

Let $w_1 = w_{\pi,P,\phi}, w_2 = w_{\rho,R,\psi} \in T_{n,k}$. As mentioned several times, in order for the product $w_1 w_2$ to be nonzero in $T_{n,k}$ each element of R must lie in a different block of π. In this situation we associate to the maps w_1 and w_2 the permutation $\tau \in S_k$ defined by the condition $r_i \in \pi_{\tau(i)}$. We say that τ *interpolates* between w_2 and w_1. (More precisely, τ describes how the elements of the image of w_2 are distributed in the blocks of the partition of w_1.)

We repeat here the example from the introduction, with $w_1 = 214442$ and $w_2 = 262225$.



$w_2 = 262225$

$\tau = (123)$

$w_1 = 214442$

Definition 4.1.1. *Let $w_1, w_2 \in T_{n,k}$ and suppose that τ interpolates between w_2 and w_1. Define*

$$w_1 * w_2 := x^{\mathrm{inv}(\tau)} w_1 w_2,$$

where $\mathrm{inv}(\tau)$ is the number of inversions of τ, i.e., the number of crossings in the diagram of τ. Let $A_{n,k}(x)$ denote the algebra with the multiplication $*$.

Example 4.1.2. *Let $w_1 = 214442$ and $w_2 = 262225$ as above. Then $\tau = (123)$ and we have*

$$w_1 * w_2 = x^2 w_1 w_2 = x^2 121114.$$

Note that if we set $x = 1$ we recover the original multiplication in $A_{n,k}$.

4.2 Associativity

Proposition 4.2.1. *The $*$-multiplication is associative.*

Proof. Let $w_1 = w_{\pi,P,\phi}, w_2 = w_{\rho,R,\psi}$, and $w_3 = w_{\sigma,S,\chi} \in T_{n,k}$. Let $\tau, \nu \in S_k$ be such that τ interpolates between w_2 and w_1, and ν interpolates between w_3 and w_2. (otherwise $w_1 w_2 w_3 = 0$).

Then

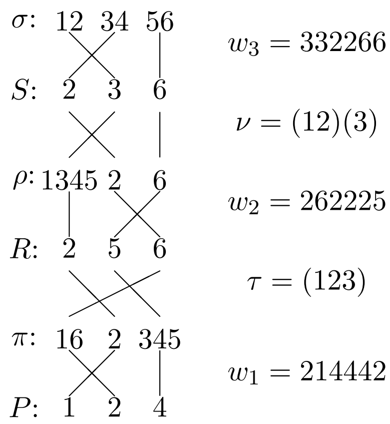
$$\begin{aligned} w_1 * (w_2 * w_3) &= w_1 * \left(x^{\mathrm{inv}(\nu)} w_{\sigma,R,\psi\nu\chi}\right) \\ &= x^{\mathrm{inv}(\tau)+\mathrm{inv}(\nu)} w_{\sigma,P,\phi\tau\psi\nu\chi}, \end{aligned}$$

while

$$\begin{aligned} (w_1 * w_2) * w_3 &= \left(x^{\mathrm{inv}(\tau)} w_{\rho,P,\phi\tau\psi}\right) * w_3 \\ &= x^{\mathrm{inv}(\tau)+\mathrm{inv}(\nu)} w_{\sigma,P,\phi\tau\psi\nu\chi}. \end{aligned}$$

□

The associativity of the $*$-multiplication is most obvious when we draw the maps in diagram form, as in the example below.



The power of x is $\mathrm{inv}(\tau) + \mathrm{inv}(\nu)$ whether we first multiply $w_1 * w_2$, or $w_2 * w_3$.

Remark 4.2.2. *We can replace* inv *by any statistic on S_k and the multiplication will still be associative.*

4.3 Gram matrix

For $w \in T_{n,k}$, let $w(x)$ denote the matrix corresponding to left $*$-multiplication by w. Note that $w(x)$ has the same support as w; the only difference being that some of the non-zero entries are positive powers of the parameter x. Let $M_{n,k}(x)$ denote the Gram matrix for $A_{n,k}(x)$, in which the i, j entry is the trace of the product $w_i(x)w_j(x)$. The following result gives a formula for the entries of $M_{n,k}(x)$.

Proposition 4.3.1.

$$(M_{n,k}(x))_{i,j} = \begin{cases} S(n,k)k!x^{\text{inv}(\tau)+\text{inv}(\kappa)} & \textit{if } w_iw_j \textit{ has } k \textit{ fixed points, } \tau \textit{ interpolates} \\ & \textit{between } w_j \textit{ and } w_i, \textit{ and } \kappa \textit{ interpolates} \\ & \textit{between } w_i \textit{ and } w_j, \textit{ and} \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. The proof is similar to that of the formula for the entries of $M_{n,k}$, the Gram matrix of the non-deformed algebra $A_{n,k}$. As in that case, $(M_{n,k}(x))_{i,j} = 0$ unless there exist r and s such that

$$\begin{aligned} w_iw_jw_r &= w_iw_s = w_r, \text{ and} \\ w_jw_iw_s &= w_jw_r = w_s. \end{aligned}$$

So w_iw_j fixes w_r and w_jw_i fixes w_s. But an element $w \in T_{n,k}$ fixes some other $w' \in T_{n,k}$ if and only if w fixes all k elements of the image of w'. So $(M_{n,k}(x))_{i,j} = 0$ unless w_iw_j and w_jw_i both have k fixed points. (Recall that these two requirements imply each other.)

Now, given such w_i and w_j, what is $\text{tr}(w_i(x)w_j(x))$? Since w_iw_j and w_jw_i both have k fixed points, there must exist $\tau, \kappa \in S_k$ such that τ interpolates between w_j and w_i and κ interpolates between w_i and w_j.

$$\begin{aligned} (M_{n,k}(x))_{i,j} &= \sum_{r=1}^{|T_{n,k}|} (w_i(x)w_j(x))_{r,r} \\ &= \sum_{r=1}^{|T_{n,k}|} \sum_{s=1}^{|T_{n,k}|} (w_i(x))_{r,s}(w_j(x))_{s,r} \\ &= \sum_{\substack{(r,s) \text{ s.t.} \\ w_iw_s = w_r \\ w_jw_r = w_s}} (w_i(x))_{r,s}(w_j(x))_{s,r} \\ &= \sum_{\substack{(r,s) \text{ s.t.} \\ w_iw_jw_r = w_r \\ w_jw_iw_s = w_s}} x^{inv(\tau)}x^{inv(\kappa)} \qquad \left(\begin{array}{l} \text{since } \text{Im}(w_r) = \text{Im}(w_i) \\ \text{and } \text{Im}(w_s) = \text{Im}(w_j) \end{array} \right) \\ &= x^{inv(\tau)+inv(\kappa)} (\#(w_r, w_s) \text{ s.t. } w_iw_jw_r = w_r \text{ and } w_jw_iw_s = w_s) \end{aligned}$$

w_s is determined by w_r, since $w_s = w_jw_r$, so

$$\begin{aligned} (M_{n,k}(x))_{i,j} &= x^{inv(\tau)+inv(\kappa)} (\#w_r \text{ s.t. } w_iw_jw_r = w_r) \\ &= x^{inv(\tau)+inv(\kappa)} (\#w \in T_{n,k} \text{ s.t. } \text{Im}(w) = \text{Im}(w_i)) \\ &= S(n,k)k!x^{\text{inv}(\tau)+\text{inv}(\kappa)} \end{aligned}$$

□

As before we normalize $M_{n,k}(x)$ by dividing by $S(n,k)k!$.
For example,

$$M_{3,2}(x) = \begin{pmatrix}
0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & x & 0 & 0 \\
0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & x \\
1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & x & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & x & 0 & x & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & x^2 & 0 & 0 & 0 & 0 & 0 & 0 & x \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & x & 0 & x & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & x & 0 & x & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 1 & x & 0 & 0 & 0 & 0 & 0 & x^2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & x & 0 & 0 & 0 & 0 & 0 & 1 & 1
\end{pmatrix}$$

Recall in Chapters 2 and 3 we were concerned with the eigenvalues of the Gram matrix. Unfortunately, the eigenvalues of $M_{n,k}(x)$ are in general not nice, and in fact the characteristic polynomial has large irreducible factors, as can be seen in the following table.

n	k	$\det(M_{n,k}(x))$	$\text{char}(M_{n,k}(x))$
2	2	1	$(-1+\lambda)^2$
2	1	0	$(-2+\lambda)\lambda$
3	3	-1	$(-1+\lambda)^5(1+\lambda)$
3	2	$(-1+x)^6(1+x)^6$	$\big((1-x)^2+\lambda(1-x)^2-\lambda^2(2+x)-\lambda^3\big)$
			$\cdot\big(-(1+x)^2+\lambda(1+x)^2+\lambda^2(2-x)-\lambda^3\big)$
			$\cdot\big((1+x)^2+\lambda(1+x)^2-\lambda^2(2-x)-\lambda^3\big)$
			$\cdot\big(-(1-x)^2+\lambda(1-x)^2+\lambda^2(2+x)-\lambda^3\big)$
			$\cdot\big((1-x)^2-3\lambda(2+x^2)+\lambda^2(5+x^2)-\lambda^3\big)$
			$\cdot\big((1+x)^2-3\lambda(2+x^2)+\lambda^2(5+x^2)-\lambda^3\big)$
3	1	0	$(-3+\lambda)\lambda^2$
4	4	-1	$(-1+\lambda)^{17}(1+\lambda)^7$
4	3	0	$\lambda^{48}\cdot$(degree 4 in λ) $\cdot$ (degree 4 in λ) $\cdot$ (degree 8 in λ)
			$\cdot$(degree 12 in λ) $\cdot$ (degree 12 in λ) $\cdot$ (degree 56 in λ)
4	2	0	$\lambda^{12}(-1-x+\lambda)(1-x+\lambda)(-1+x+\lambda)(1+x+\lambda)$
			$\cdot\big((1-x)^2-2\lambda(2+x+x2)+\lambda^2\big)$
			$\cdot\big((1+x)^2-2\lambda(2-x+x2)+\lambda^2\big)$
			$\cdot$(degree 4 in λ) $\cdot$ (degree 4 in λ)
			$\cdot$(degree 12 in λ) $\cdot$ (degree 12 in λ)
			$\cdot$(degree 16 in λ) $\cdot$ (degree 16 in λ)
4	1	0	$(-4+\lambda)\lambda^3$

Note: The paired irreducible polynomials appearing above are equal up to replacing x by $-x$.

In the next table we compare dimensions of the corresponding nullspaces.

n	k	$M_{n,k}$	$M_{n,k}(x)$
2	2	0	0
2	1	1	1
3	3	0	0
3	2	5	0
3	1	2	2
4	4	0	0
4	3	55	48
4	2	15	12
4	1	3	3

In the case of $M_{n,k}(x)$ the listed dimensions are for *generic* x.

Note that nullspace of $M_{n,k}(x)$ appears to be smaller in general than that of $M_{n,k}$. This means that our deformation is having the desired effect: it is reducing the size of the radical. To make this more precise, we employ the Munn algebra structure of $A_{n,k}$.

4.4 $A_{n,k}(x)$ is a Munn algebra

Recall the following fact from Chapter 2.

Fact 4.4.1. *$A_{n,k}$ is isomorphic to the Munn matrix algebra $\mathcal{M}(\mathbb{C}S_k; \binom{[n]}{k}, S([n],k); \Pi)$. The elements of this semigroup are $\binom{n}{k} \times S(n,k)$ matrices over $\mathbb{C}S_k$. Multiplication is defined by $w_1 \cdot w_2 := w_1 \Pi w_2$, where Π is the $S(n,k) \times \binom{n}{k}$ sandwich matrix*

$$(\Pi)_{\pi,P} = \begin{cases} \tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \leq i \leq k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Note that the non-zero entries of $\Pi_{n,k}$ are precisely the permutations τ that arise in the $*$ multiplication. Hence our second deformation preserves the Munn matrix algebra structure.

Proposition 4.4.2. *$A_{n,k}(x)$ is isomorphic to the Munn matrix algebra $\mathcal{M}(\mathbb{C}S_k; \binom{[n]}{k}, S([n],k); \Pi(x))$, where $\Pi(x)$ is the $S(n,k) \times \binom{n}{k}$ sandwich matrix defined by*

$$(\Pi(x))_{\pi,P} = \begin{cases} x^{inv(\tau)}\tau & \text{if } p_i \in \pi_{\tau(i)}, 1 \leq i \leq k \\ 0 & \text{otherwise.} \end{cases}$$

Example 4.4.3. *For $n = 4, k = 3$ we have*

$$\Pi(x) = \begin{pmatrix} (id) & (id) & 0 & 0 \\ 0 & (id) & (id) & 0 \\ 0 & 0 & (id) & (id) \\ (id) & 0 & 0 & x^2(123) \\ (id) & 0 & x(23) & 0 \\ 0 & (id) & 0 & x(12) \end{pmatrix},$$

where as before the ordering on the columns is $123, 124, 134, 234$, and the ordering on the rows is $1|2|34, 1|23|4, 12|3|4, 14|2|3, 1|24|3, 13|2|4$. Now let $w_1 = 2142$ and $w_2 = 4223$. Then

$$w_1 * w_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & (12) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} (id) & (id) & 0 & 0 \\ 0 & (id) & (id) & 0 \\ 0 & 0 & (id) & (id) \\ (id) & 0 & 0 & x^2(123) \\ (id) & 0 & x(23) & 0 \\ 0 & (id) & 0 & x(12) \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (132) & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= x^2 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & (12) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} = x^2 2114$$

Note that showing that $A_{n,k}(x)$ is a Munn matrix algebra provides another proof that the $*$-multiplication is associative.

When $k = 1$ the parameter x does not appear at all, and the semisimple part of $A_{n,1}$ is only one-dimensional. For the remainder of this chapter we shall assume $k > 1$.

4.5 The Structure of $A_{n,k}(x)/\sqrt{A_{n,k}(x)}$

Since we want our Munn matrix algebra to be semisimple, it is natural to ask in what way the semisimplicity of $A_{n,k}(x)$ depends on the sandwich matrix $\Pi_{n,k}(x)$. Recall that the following result of Clifford and Preston ([3], Theorem 5.19) provides an answer to this question.

Theorem 4.5.1. *(Clifford and Preston) A Munn matrix algebra of the form* $\mathcal{A} = \mathcal{M}(\mathbb{C}G; m, n; \Pi)$ *is semisimple if and only if* Π *is non-singular, i.e., if and only if* $m = n$ *and* Π *is a unit in the ring of* $m \times m$ *matrices over* $\mathbb{C}G$*.*

Note in particular that for semisimplicity we need the matrices to be square. But our sandwich matrix is $S(n,k) \times \binom{n}{k}$. What can we do?

Define the *rank* of Π to be the largest non-singular minor of Π. (So, in particular, $\mathrm{rank}(\Pi) \leq \min(m,n)$.) We also need the following technical condition, known as *suitability.*

Definition 4.5.2. *Let* P *be an* $n \times m$ *matrix over* A *with rank* r*. Let* R *and* S *be permutation matrices over* A *such that*

$$RPS = \begin{pmatrix} M & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where M *is an invertible* $r \times r$ *submatrix of* P*, and let*

$$Q = S \begin{pmatrix} M^{-1} & 0 \\ 0 & 0 \end{pmatrix} R.$$

Then we say that P *is* suitable *if* $PQP - P \in (\sqrt{A})_{n \times m}$*.*

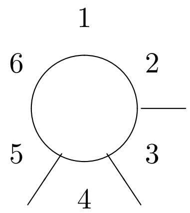
Rank and suitability are intimately related to the size of $\sqrt{\mathcal{A}}$, as the following result of McAlister [10] shows.

Theorem 4.5.3. *(McAlister) Let* $\mathcal{A} = \mathcal{M}(A; m, n; \Pi)$ *be a Munn matrix algebra, and let* Π *be suitable of rank* r*. Then* $\mathcal{A}/\sqrt{\mathcal{A}} \cong \left(A/\sqrt{A}\right)_r$*, the algebra of all* $r \times r$ *matrices with entries in* $A/\sqrt{A}$*.*

The suitability condition does not hold for the undeformed algebra $A_{n,k}$. But what about for $A_{n,k}(x)$, is x is generic? Our first step is to determine the rank of $M_{n,k}(x)$, by finding a maximal non-singular submatrix.

Definition 4.5.4. *A set partition π of $[n]$ is* cyclically contiguous *if the blocks of π are intervals, with the possible exception of the first block, which may be of the form $\{1, 2, \ldots, i\} \cup \{j, j+1, \ldots, n\}$, i.e., the union of an initial segment and a terminal segment. If the first block is also an interval, we say that π is* contiguous. *(Note that contiguous implies cyclically contiguous, not the other way around.)*

We use the term "cyclically contiguous" for such a partition π because if we think of the elements of $[n]$ as being arranged in (clockwise) order around a circle, then in a sense *all* of the blocks of π are intervals. For example, $\pi = 1256|3|4$ is cyclically contiguous, as shown in the following diagram.

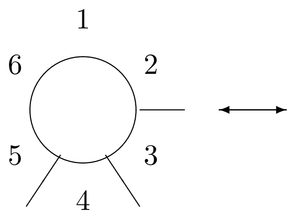


$\pi = 1256|3|4$

Proposition 4.5.5. *For $n \geq k > 1$ the number of cyclically contiguous partitions of $[n]$ into k blocks is $\binom{n}{k}$.*

Proof. We exhibit a bijection between cyclically contiguous partitions of $[n]$ into k blocks and k-subsets of $[n]$. Given a cyclically contiguous partition $\pi = \{\pi_1, \pi_2, \ldots, \pi_k\}$ define the set $P(\pi) = \{p_1, p_2, \ldots, p_k\}$ where p_i is the "clockwise minima" of π_i. Call $P(\pi)$ the "set of dividers" of π. $P(\pi)$ is a k-subset of $[n]$, and this process is clearly reversible: given a subset $P \subseteq [n], |P| = k$, write elements of $[n]$ in clockwise order around a circle and put a divider before each element of P. □

The bijection described above is easily visualized if we again think of partitioning the elements of $[n]$ arranged around a circle.



$P(\pi) = \{3, 4, 5\}$

$\pi = 1256|3|4$

Note that in the case π is actually *contiguous*, rather than merely cyclically contiguous, then $p_i = \min(\pi_i)$. On the other hand, if $\pi_1 = 1, 2, \ldots, i, j, j+1, \ldots, n$, then $p_i = \min(\pi_{i+1}), 1 \leq i \leq k-1$, and $p_k = j$. There are $\binom{n-1}{k-1}$ contiguous partitions, corresponding to the k-subsets of $[n]$ containing the element 1.

Let $\Pi^c_{n,k}(x)$ be the $\binom{n}{k}\times\binom{n}{k}$ submatrix of $\Pi_{n,k}(x)$ consisting of the rows corresponding to cyclically contiguous partitions. For example,

$$\Pi^c_{5,3}(x) = \begin{pmatrix} I & I & I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & I & I & I & I & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & I & I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I & I & 0 & I & I & 0 & 0 \\ 0 & 0 & 0 & 0 & I & I & 0 & I & I & 0 \\ 0 & 0 & 0 & 0 & 0 & I & 0 & 0 & I & I \\ I & 0 & 0 & 0 & 0 & 0 & x^2\alpha & x^2\alpha & 0 & 0 \\ I & I & 0 & 0 & 0 & 0 & 0 & x^2\alpha & x^2\alpha & 0 \\ 0 & I & 0 & I & 0 & 0 & 0 & 0 & x^2\alpha & x^2\alpha \\ 0 & 0 & 0 & I & 0 & 0 & I & 0 & 0 & x^2\alpha \end{pmatrix},$$

where $I = I_3$ and $\alpha = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = (123) \in S_3$. As we shall do for the remainder of this chapter, we are ordering the rows and columns lexicographically (the rows by the divider sets $P(\pi)$).

Note that the only non-zero, non-identity element that can appear in $\Pi^c_{n,k}(x)$ is the k-cycle α. Since α commutes with I and with itself, the determinant of $\Pi^c_{n,k}(x)$ is well-defined.

Proposition 4.5.6. *Let $\Pi^c_{n,k}(x)$ be as above. Then*

$$\det(\Pi^c_{n,k}(x)) = \Big((-1)^k(1 - x^{k(k-1)})\Big)^{\binom{n-1}{k}}.$$

Proof. We shall show that $\Pi^c_{n,k}(x)$ is equivalent via elementary row operations to an upper-triangular block matrix $\bar{\Pi}^c_{n,k}(x)$ of the form

$$\bar{\Pi}^c_{n,k}(x) = \left(\begin{array}{ccc|ccc} I & & * & & & \\ & \ddots & & & * & \\ 0 & & I & & & \\ \hline & & & x^{k-1}\alpha + (-1)^k I & & * \\ & 0 & & & \ddots & \\ & & & 0 & & x^{k-1}\alpha + (-1)^k I \end{array}\right).$$

More precisely,

$$\left(\bar{\Pi}^c_{n,k}(x)\right)_{\pi,S} = \begin{cases} I_k & \text{if } s_i \in \pi_i, 1 \le i \le k, \text{ and } P(\pi) \le S \\ x^{k-1}\alpha + (-1)^k I & \text{if } s_i \in \pi_{i+1}, 1 \le i \le k-1, s_k \in \pi_1, \text{ and } P(\pi) \le S \\ 0 & \text{otherwise.} \end{cases}$$

As in the example we are using the lexicographic order on the k-subsets of $[n]$. The added condition $P(\pi) \le S$ forces $\bar{\Pi}^c_{n,k}(x)$ to be upper triangular.

It is a fact that if the blocks of a block matrix commute with each other, the determinant of the matrix can be computed by taking the determinant of the matrix whose entries are the determinants of the corresponding blocks. Thus we have

$$\begin{aligned}
\det\Pi^c_{n,k}(x) &= \det\bar{\Pi}^c_{n,k}(x) \\
&= (\det I)^{\binom{n-1}{k-1}}\left(\det(x^{k-1}\alpha+(-1)^k I)\right)^{\binom{n-1}{k}} \\
&= \left(x^{k(k-1)}(-1)^{k-1}+(-1)^{k^2}\right)^{\binom{n-1}{k}} \\
&= \left((-1)^k(1-x^{k(k-1)})\right)^{\binom{n-1}{k}}. \quad (k \text{ and } k^2 \text{ have the same parity.})
\end{aligned}$$

To complete the proof we need to show how to convert $\Pi^c_{n,k}(x)$ into $\bar{\Pi}^c_{n,k}(x)$.

Claim 1: The first $\binom{n-1}{k-1}$ rows of $\Pi^c_{n,k}(x)$ and $\bar{\Pi}^c_{n,k}(x)$ are identical.

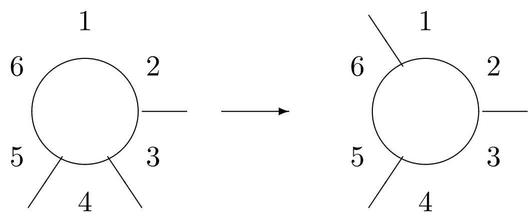
Proof of Claim 1: The first $\binom{n-1}{k-1}$ rows correspond to the contiguous partitions. Let π be a contiguous partition, and let S be a subset with $P(\pi) > S$. Then there exists l such that $p_j = s_j, 1 \le j \le l$, while $p_{l+1} > s_{l+1}$. But then $\{s_1, s_2, \ldots, s_{l+1}\} \subseteq \bigcup_{j=1}^{l} \pi_j = \{1, 2, \ldots, p_{l+1}-1\}$. (Recall that in the contiguous case $p_i = \min(\pi_i)$.) So by the Pigeonhole Principle at least two of the s_i are in the same block of π and hence $\left(\Pi^c_{n,k}(x)\right)_{\pi,S} = 0$. Thus the extra condition $P(\pi) \le S$ is vacuous in the case π is contiguous.

Note also that π contiguous precludes the possibility that $s_i \in \pi_{i+1}, 1 \le i \le k-1, s_k \in \pi_1$, since in that case $s_k > s_1$, while every element of π_1 is less than every element of π_2.

So $\left(\Pi^c_{n,k}(x)\right)_{\pi,S} = \left(\bar{\Pi}^c_{n,k}(x)\right)_{\pi,S}$ for π a contiguous partition, thus proving Claim 1.

What about the last $\binom{n-1}{k}$ rows?

For π not contiguous (but still cyclically contiguous), denote by $\pi^{(j)}$ the contiguous partition with divider set $(P(\pi)\backslash p_j) \cup \{1\}$. In the example below, $\pi = 1256|3|4$ and $P(\pi) = \{3,4,5\}$. To obtain $\pi^{(2)}$ we remove the second divider $p_2 = 4$ and add the divider 1, thus giving $P(\pi^{(2)}) = \{1,3,5\}$ and so $\pi^{(2)} = 12|34|56$.



$\pi = 1256|3|4$ $\qquad$ $\pi^{(2)} = 12|34|56$

$P(\pi) = \{3,4,5\}$ $\qquad$ $P(\pi^{(2)}) = \{1,3,5\}$

For π not contiguous (but still cyclically contiguous) we define the following row operation.

$$R_\pi \longrightarrow R_\pi - (-1)^k \sum_{j=1}^{k} (-1)^j R_{\pi^{(j)}},$$

where R_π is the row of $\Pi^c_{n,k}(x)$ indexed by π, and

Claim 2: These row operations convert $\Pi^c_{n,k}(x)$ into $\bar{\Pi}^c_{n,k}(x)$.

Proof of Claim 2: Suppose that π is not contiguous, and let S be a k-subset of $[n]$. We have three cases:

Case 1. $s_i \in \pi_i, 1 \le i \le k$. Note that this implies $s_1 \in \{1, 2, \ldots, p_1 - 1\}$, the "front half" of π_1. We have $\left(\Pi^c_{n,k}(x)\right)_{\pi,S} = I$. After the row operations, the (π, S) entry is

$$\begin{aligned} I - (-1)^k \sum_{j=1}^{k} (-1)^j \left(R_{\pi^{(j)}}\right)_S &= I - (-1)^k (-1)^k \left(R_{\pi^{(k)}}\right)_S \\ &= I - I \\ &= 0, \end{aligned}$$

since the only way S can be distributed in the blocks of $\pi^{(j)}$ is to remove the last divider p_k of π, thus merging π_k and the "back half" of π_1 (which contained no element of S). Removing any other divider merges two blocks both of which contain an element of S.

Case 2. $s_i \in \pi_{i+1}, 1 \le i \le k-1, s_k \in \pi_1$. Note that this implies $s_k \in \{p_k, p_k + 1, \ldots, n\}$, the "back half" of π_1. We have $\left(\Pi^c_{n,k}(x)\right)_{\pi,S} = x^{k-1}\alpha$. After the row operations, the (π, S) entry is

$$\begin{aligned} x^{k-1}\alpha - (-1)^k \sum_{j=1}^{k} (-1)^j \left(R_{\pi^{(j)}}\right)_S &= x^{k-1}\alpha - (-1)^k (-1)^1 \left(R_{\pi^{(1)}}\right)_S \\ &= x^{k-1} + (-1)^k I, \end{aligned}$$

since the only way S can be distributed in the blocks of $\pi^{(j)}$ is to remove the first divider p_1 of π, thus merging π_2 (which contains s_1) and the front half $\{1, 2, \ldots, p_1 - 1\}$ of π_1 (which contains no element of S). Removing any other divider merges two blocks both of which contain an element of S.

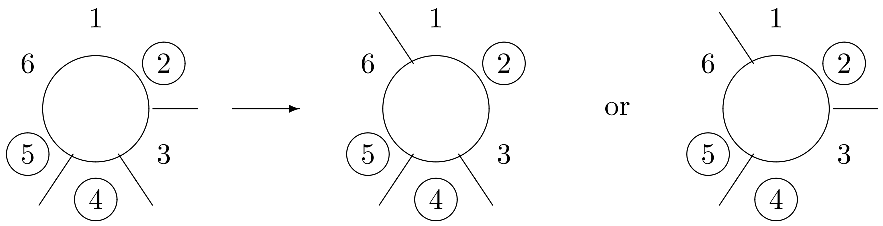
Case 3. **S is not distributed in the blocks of π, so that** $\left(\Pi^c_{n,k}(x)\right)_{\pi,S} = 0$. This can happen in several ways.

Case 3.1. **π_1 contains two elements of S, one in the front half, one in the back half, and π_i contains exactly one element of S for all $2 \le i \le k$ except π_l contains no element of S, for precisely one $l, 2 \le l \le k$.** Note that since π is not contiguous, $\pi_l = \{p_{l-1}, p_{l-1}+1, \ldots, p_l - 1\}$. After the row operations, the (π, S) entry is

$$\begin{aligned} 0 - (-1)^k \sum_{j=1}^{k} (-1)^j \left(R_{\pi^{(j)}}\right)_S &= (-1)^{k+1} \left((-1)^{l-1} (R_{\pi^{(l-1)}})_S + (-1)^l (R_{\pi^{(l)}})_S \right) \\ &= (-1)^{k+1} \left((-1)^{l-1} I + (-1)^l I \right) \\ &= 0 \end{aligned}$$

since the only way S can be distributed in the blocks of $\pi^{(j)}$ is to either remove the divider p_{l-1}, thus merging π_l and π_{l-1}, or remove the divider p_l, thus merging π_l and π_{l+1} (or in the case $l = k$, merging π_k and the back half of π_1).

In the example below, $\pi = 1256|3|4$ and $S = \{2, 4, 5\}$ (circled). The block π_1 contains two elements of S, π_2 contains none, and π_2 contains 1. In order for S to be distributed in the blocks, we must either remove the divider $p_1 = 3$, merging π_1 and π_2, or the divider $p_2 = 4$, merging π_2 and π_3.



$\pi = 1256|3|4$ $\quad$ $\pi^{(1)} = 123|4|56$ $\quad$ $\pi^{(2)} = 12|34|56$

$P(\pi) = \{3, 4, 5\}$ $\quad$ $P(\pi^{(1)}) = \{1, 4, 5\}$ $\quad$ $P(\pi^{(2)}) = \{1, 3, 5\}$

Case 3.2. All other possibilities. In this case there is no way for S to be distributed in the blocks of $\pi^{(j)}$, no matter which divider we remove. Some $\pi_i, 2 \leq i \leq k$, or the front or back half of π_1, contains at least two elements of S. $\left(\Pi^c_{n,k}(x)\right)_{\pi,S} = 0$. and after the row operation, the (π, S) entry is

$$0 - (-1)^k \sum_{j=1}^{k} (-1)^j \left(R_{\pi^{(j)}}\right)_S = 0.$$

Putting all of the cases together, we see that the row operations

$$R_\pi \longrightarrow R_\pi - (-1)^k \sum_{j=1}^{k} (-1)^j R_{\pi^{(j)}},$$

- have no effect on the first $\binom{n-1}{k-1}$ rows of $\Pi^c_{n,k}(x)$,
- convert each I in the bottom $\binom{n-1}{k}$ rows to 0 (Case 1),

- convert each $x^{k-1}\alpha$ in the bottom $\binom{n-1}{k}$ rows to $x^{k-1}\alpha + (-1)^k I$ (Case 2),
- preserve all 0's (Case 3).

Thus $\Pi^c_{n,k}(x)$ is row equivalent to $\bar{\Pi}^c_{n,k}(x)$ and we are done. □

So in particular, if x is not a $k(k-1)^{\text{st}}$ root of unity, $\Pi^c_{n,k}(x)$ is non-singular as a matrix over $\mathbb{C}$. But is it non-singular in the sense defined above? In other words, does it have an inverse as a matrix with entries in $\mathbb{C}S_k$?

Recall that for a matrix A over a *field* we can compute the inverse matrix A^{-1} using cofactors:

$$\left(A^{-1}\right)_{i,j} = \frac{(-1)^{i+j}\det M_{j,i}}{\det A},$$

where $M_{i,j}$ is the submatrix of A obtained by deleting row j and column i. If $\det A$ is a unit this procedure works over a commutative ring, since the rest is all multiplication and addition.

In our case we have

$$\det \Pi^c_{n,k}(x) = \left(x^{k-1}\alpha + (-1)^k I\right)^{\binom{n-1}{k}},$$

so we need $x^{k-1}\alpha + (-1)^k I$ to be invertible in the (commutative) subalgebra of $\mathbb{C}S_k$ generated by α.

Claim:

$$\left(x^{k-1}\alpha + (-1)^k I\right)^{-1} = \frac{1}{(-1)^k(1-x^{k(k-1)})}\sum_{i=0}^{k-1}(-1)^{i(k-1)}x^{i(k-1)}\alpha^i.$$

Proof of Claim:

$$\begin{aligned}
\left(x^{k-1}\alpha + (-1)^k I\right)\sum_{i=0}^{k-1}(-1)^{i(k-1)}x^{i(k-1)}\alpha^i &= \sum_{i=0}^{k-1}(-1)^{i(k-1)}x^{(i+1)(k-1)}\alpha^{i+1} + (-1)^{i(k-1)+k}x^{i(k-1)}\alpha^i \\
&= \sum_{i=1}^{k}(-1)^{(i-1)(k-1)}x^{i(k-1)}\alpha^i + \sum_{i=0}^{k-1}(-1)^{i(k-1)+k}x^{i(k-1)}\alpha^i \\
&= (-1)^{(k-1)^2}x^{k(k-1)}I + (-1)^k I \\
&\quad + \sum_{i=1}^{k-1}\left((-1)^{ik-i-k+1} + (-1)^{ik-i+k}\right)x^{i(k-1)}\alpha^i \\
&= (-1)^k(1-x^{k(k-1)})I,
\end{aligned}$$

since $(k-1)^2$ and k have opposite parity. So we have proved the following proposition.

Proposition 4.5.7. *For x not a $k(k-1)^{st}$ root of unity, $\Pi^c_{n,k}(x)$ is non-singular and hence $\Pi_{n,k}(x)$ has full rank $\binom{n}{k}$.*

Recall now the definition of suitability.

Definition 4.5.8. *Let P be an $n \times m$ matrix over A with rank r. Let R and S be permutation matrices over A such that*

$$RPS = \begin{pmatrix} M & P_{12} \\ P_{21} & P_{22} \end{pmatrix},$$

where M is an invertible $r \times r$ submatrix of P, and let

$$Q = S \begin{pmatrix} M^{-1} & 0 \\ 0 & 0 \end{pmatrix} R,$$

Then P is suitable *if $PQP - P \in (\sqrt{A})_{n \times m}$.*

Suppose that $n \times m$ sandwich matrix P ($n > m$) has full rank, i.e. that there exists permutation matrices R and S such that

$$RPS = \begin{pmatrix} M \\ P_{21} \end{pmatrix},$$

where M is an invertible $m \times m$ submatrix of P. Let $Q = S \begin{pmatrix} M^{-1} & 0 \end{pmatrix} R$. Then

$$\begin{aligned} PQP - P &= R^{-1} \begin{pmatrix} M \\ P_{21} \end{pmatrix} S^{-1} S \begin{pmatrix} M^{-1} & 0 \end{pmatrix} R R^{-1} \begin{pmatrix} M \\ P_{21} \end{pmatrix} S^{-1} - R^{-1} \begin{pmatrix} M \\ P_{21} \end{pmatrix} S^{-1} \\ &= R^{-1} \begin{pmatrix} M \\ P_{21} \end{pmatrix} \begin{pmatrix} M^{-1} & 0 \end{pmatrix} \begin{pmatrix} M \\ P_{21} \end{pmatrix} S^{-1} - R^{-1} \begin{pmatrix} M \\ P_{21} \end{pmatrix} S^{-1} \\ &= R^{-1} \begin{pmatrix} M \\ P_{21} \end{pmatrix} S^{-1} - R^{-1} \begin{pmatrix} M \\ P_{21} \end{pmatrix} S^{-1} \\ &= 0. \end{aligned}$$

So in the case that P has full rank the suitability condition is automatically satisfied.

Thus by Proposition 4.5.7 and McAlister's result we have

Theorem 4.5.9. *For $k > 1$ and generic x, $A_{n,k}(x)/\sqrt{A_{n,k}(x)} \cong (\mathbb{C}S_k)_{\binom{n}{k}}$.*

Note that the rank of Π cannot be any larger than its width $\binom{n}{k}$. So no deformation of $A_{n,k}$ that preserves the Munn matrix algebra structure can be any more semisimple than $A_{n,k}(x)$.

Corollary 4.5.10. *For $k > 1$,*

$$\dim \left(\sqrt{A_{n,k}(x)} \right) = \left(S(n,k) - \binom{n}{k} \right) \binom{n}{k} k!.$$

Since this dimension formula is relatively simple one might hope to find a nice combinatorial basis for $\sqrt{A_{n,k}(x)}$, as Garsia and Reutenauer did in [5] for Solomon's Descent Algebra.

4.6 The y deformation

Let $A^c_{n,k}(x)$ be the subalgebra of $A_{n,k}(x)$ spanned by the maps $T^c_{n,k}$ whose partitions are cyclically contiguous. $A^c_{n,k}(x)$ is also a Munn matrix algebra, with sandwich matrix $\Pi^c_{n,k}(x)$. As mentioned in Chapter 2, Grood [6] has generalized the classical Specht-module construction for the Symmetric Group to describe the irreducible modules of the Rook Monoid R_n. Recall that, R_n has a composition series

$$R_n = J_n \supseteq J_{n-1} \supseteq \ldots \supseteq J_1 \supseteq J_0 = 0,$$

and that we have defined $R_{n,k} := J_k/J_{k-1}$. It is not difficult to see that $B_{n,k} := \mathbb{C}R_{n,k}$ is isomorphic to the Munn matrix algebra $\mathcal{M}(\mathbb{C}S_k; \binom{n}{k}, \binom{n}{k}; I) = (\mathbb{C}S_k)_{\binom{n}{k}}$, the algebra of $\binom{n}{k} \times \binom{n}{k}$ matrices with entries in the group algebra $\mathbb{C}S_k$. In what follows we derive a deformation of $A^c_{n,k}$, written $A^c_{n,k}(\mathbf{y})$, so that there are canonical isomorphisms $A^c_{n,k}(\mathbf{1}) \cong A^c_{n,k}$ and $A^c_{n,k}(\mathbf{0}) \cong B_{n,k}$.

4.6.1 Definition

Let $w_1 = w_{\pi,S,\sigma}, w_2 = w_{\nu,T,\phi} \in T^C_{n,k}$. As we have seen before, for the product w_1w_2 to be nonzero in $T_{n,k}$ each element of T must lie in a different block of π. In other words, each block π_i of π contains exactly one element of T. For this deformation we shall pay attention to how close that element is to the clockwise minimal element of π_i.

Recall that to a cyclically contiguous partition π we associate a corresponding divider set $P(\pi) = \{p_1, \ldots, p_k\}$. ($P$ consists of the clockwise minimal elements of the blocks of π.) In what follows we will often identify a partition with its set of dividers.

For each $i, 1 \leq i \leq k$, let $\rho_i(\pi, T)$ be the difference (modulo n) between the element of T lying in π_i and the minimal element of π_i. For example, if $\pi = 156|2|34$ and $T = \{1, 2, 4\}$, then $\rho_2(\pi, T) = 2 - 2 = 0$, $\rho_3(\pi, T) = 4 - 3 = 1$, while $\rho_1(\pi, T) = 1 - 5 = -4 = 2(\text{mod } 6)$. We are now ready to define the deformation.

Definition 4.6.1. *Let $w_1 = w_{\pi,S,\sigma}, w_2 = w_{\nu,T,\phi} \in T^C_{n,k}$. Define*

$$w_1 \bullet w_2 = \begin{cases} 0 & \text{if } w_1w_2 = 0 \text{ in } T_{n,k}, \text{ and} \\ y_1^{\rho_1(\pi,T)} \cdots y_k^{\rho_k(\pi,T)} w_1w_2 & \text{otherwise.} \end{cases}$$

Let $A^c_{n,k}(\mathbf{y})$ be the complex algebra with basis $T^C_{n,k}$, where two elements multiply via $\bullet$.

Note that $A^c_{n,k}(\mathbf{1}) = A_{n,k}$, where $\mathbf{1} = (1, \ldots, 1)$.

Example 4.6.2. *Let $w_1 = 265522$ and $w_2 = 422122$. Following the notation above we have $T = \{1, 2, 4\}$ and $\pi = 156|2|34$. So*

$$w_1 \bullet w_2 = y_1^2 y_2^0 y_3^1 w_1w_2 = y_1^2 y_3 (566266).$$

Because we are sticking to our convention of always labeling the elements of a set in increasing order, to give a more explicit definition of this deformation we must consider three separate cases.

Case 1: $1 \in P(\pi)$

In this case $p_1 = 1$, and thus π_1 is the initial segment $\{1, 2, \ldots, p_2 - 1\}$. Since the elements of T are labeled in increasing order, we have $t_i \in \pi_i$, for $i = 1, \ldots, k$. So

$$w_1 \bullet w_2 := y_1^{t_1 - p_1} \cdots y_k^{t_k - p_k} w_1 w_2.$$

Case 2.1: $1 \notin P(\pi), t_1 \in \pi_1$

In this case π_1 is the union of the terminal segment $\{p_k, \ldots, n\}$ and the initial segment $\{1, 2, \ldots, p_1 - 1\}$. As in Case 1 we have $t_i \in \pi_i$, for $i = 1, \ldots, k$. So

$$w_1 \bullet w_2 := y_1^{t_1 - p_k + n} y_2^{t_2 - p_1} \cdots y_k^{t_k - p_{k-1}} w_1 w_2.$$

Case 2.2: $1 \notin P(\pi), t_1 \notin \pi_1$

As in Case 2.1 π_1 is the union of the terminal segment $\{p_k, \ldots, n\}$ and the initial segment $\{1, 2, \ldots, p_1 - 1\}$. But now we have $t_i \in \pi_{i+1}$, for $i = 1, \ldots, k-1$, and $t_k \in \pi_1$. So

$$w_1 \bullet w_2 := y_1^{t_k - p_k} y_2^{t_1 - p_1} \cdots y_k^{t_{k-1} - p_{k-1}} w_1 w_2.$$

Recall that for $P, S \in \binom{[n]}{k}$ and $\sigma \in S_k$, we denote by $w_{P,S,\sigma}$ the map in $R_{n,k}$ sending p_i to $s_{\sigma(i)}$.

Proposition 4.6.3. *The mapping* $w_{\pi,S,\sigma} \mapsto w_{P,S,\sigma}$ *induces an algebra isomorphism between* $A^c_{n,k}(\mathbf{0})$ *and* $R_{n,k}$.

Proof. Let $w_{\pi,S,\sigma}, w_{\nu,T,\phi} \in T^C_{n,k}$, and let $w_{P,S,\sigma}, w_{N,T,\phi}$ be the corresponding maps in $R_{n,k}$.

$$w_{\pi,S,\sigma} \bullet w_{\nu,T,\phi} = \begin{cases} y_1^{\rho_1(\pi,T)} \cdots y_k^{\rho_k(\pi,T)} w_{\pi,S,\sigma} w_{\nu,T,\phi} & \text{if } t_i \in \pi_{\tau(i)} \text{ for some } \tau \in S_k \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

If all the y_i are zero however, to get a non-zero product we need $\rho_i(\pi, T) = 0$ for all $i = 1, \ldots, k$. This only happens when T consists of precisely the minimal elements of the blocks of π, i.e., when $T = P$. So we have

$$w_{\pi,S,\sigma} \bullet w_{\nu,T,\phi} = \begin{cases} w_{\pi,S,\sigma} w_{\nu,T,\phi} = w_{\nu,S,\sigma\phi} & \text{if } T = P \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

On the other hand,

$$w_{P,S,\sigma} w_{N,T,\phi} = \begin{cases} w_{N,S,\sigma\phi} & \text{if } T = P \text{, and} \\ 0 & \text{otherwise.} \end{cases}$$

Thus the mapping $w_{\pi,S,\sigma} \mapsto w_{P,S,\sigma}$ between the basis elements of the two algebras commutes with multiplication.

□

4.6.2 Generic semisimplicity

Proposition 4.6.4. *$A^c_{n,k}(\mathbf{y})$ is a Munn algebra, with sandwich matrix $\Pi^c_{n,k}(\mathbf{y})$ defined by*

$$(\Pi_{n,k}(\mathbf{y}))_{\pi,T} = \begin{cases} y_1^{\rho_1(\pi,T)} \cdots y_k^{\rho_k(\pi,T)}\tau & \text{if } t_i \in \pi_{\tau(i)}, \text{ for some } \tau \in S_k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The $\mathbf{y}$-deformation of the product $w_1 \bullet w_2$ only depends on how the image of w_2 is distributed in the partition of w_1; the underlying composition of maps is the same as in $T^c_{n,k}$. Hence $A^c_{n,k}(\mathbf{y})$ is isomorphic to a Munn algebra of $\binom{n}{k}$-by-$\binom{n}{k}$ matrices with entries in $\mathbb{C}S_k$. The sandwich matrix is the sandwich matrix for $A^c_{n,k}$, with the (π, T) entry scaled by $y_1^{\rho_1(\pi,T)} \cdots y_k^{\rho_k(\pi,T)}$. □

Corollary 4.6.5. *The multiplication $\bullet$ is associative.*

4.6.3 Main result

Theorem 4.6.6. *Let $\Pi^c_{n,k}(\mathbf{y})$ be as above, and let $\alpha \in S_k$ be the k-cycle $(12\dots k)$. Then*

$$\det(\Pi^c_{n,k}(\mathbf{y})) = (-1)^{k\binom{n-1}{k}} \prod_{P \in \binom{[n]\setminus 1}{k}} \left(\left(y_1^{n+p_1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}} \right)^k - 1 \right).$$

Proof. First of all, though the entries of $\Pi^c_{n,k}(\mathbf{y})$ are in the non-commutative ring $\mathbb{C}S_k$, only (scalar multiples of) powers of α actually appear. Thus all matrix entries commute and the determinant is well-defined.

To compute the determinant we shall use a series of row operations to transform $\Pi^c_{n,k}(\mathbf{y})$ into an upper triangular matrix. These row operations will be similar to those used to compute $\det(\Pi^c_{n,k}(x))$.

Recall that the rows of $\Pi^c_{n,k}(\mathbf{y})$ are indexed by the contiguous partitions of n into k blocks. Let π be such a partition, and let P be the corresponding set of dividers.

Case 1: $1 \in P$ (in particular $p_1 = 1$)

In this case, if $T < P$ then by the Pigeonhole Principle there exists a $j, 1 \le j \le k-1$ such that two elements of T lie between p_j (inclusive) and p_{j+1}. These two elements must therefore lie in the same block of π, and so $(\Pi_{n,k}(\mathbf{y}))_{\pi,T} = 0$. If $T = P$ then $\rho_i(\pi, T) = 0$, and $t_i \in \pi$, for all $i = 1, \dots, k$. Hence $(\Pi_{n,k}(\mathbf{y}))_{\pi,P} = I_k$. So the first $\binom{n-1}{k-1}$ rows of $\Pi_{n,k}(\mathbf{y})$ are upper triangular with I_k's down the diagonal. No row operations are needed in this case.

Case 2: $1 \notin P$

Note first that the pigeonhole argument in Case 1 no longer holds. For example $T = \{1, 3, 6\} < P = \{2, 3, 4\}$ yet no two elements of T lie in the same block of π.

For $1 \le j \le k$ and $P \in \binom{[n]\setminus 1}{k}$ let $P_{(j)}$ denote the set obtained from P by removing p_j and adjoining 1. Let $\phi_i(j)$ denote the number of elements (including p_j) in the ith block of π that we pass as we move clockwise between p_j and 1. For example, if $n = 6$ and $P = \{2, 3, 4\}$, then $P_{(2)} = \{1, 2, 4\}$, and $\phi_1(2) = 3$ since we pass by $4, 5, 6 \in \pi_1$ as we move clockwise between 2 and 1. Similarly, $\phi_2(2) = 0$ (pass none) and $\phi_3(2) = 1$ (pass three).

Let R_P denote the P^{th} row of $\Pi_{n,k}(\mathbf{y})$. We perform the following row operation.

$$R_P \longrightarrow R_P - (-1)^k \sum_{j=1}^{k} (-1)^j \Big(\prod_{i=1}^{k} y_i^{\phi_i(j)} \Big) R_{P_{(j)}}$$

Let $T \in \binom{[n]}{k}$ and consider the (P, T) entry of the new matrix. We have three cases.

Case 2.1: $t_i \in \pi_i, 1 \leq i \leq k$

In particular, $t_1 \in \{1, \ldots, p_1 - 1\}$, the "front half" of π_1. Then $(\Pi_{n,k}(\mathbf{y}))_{P,T} = y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} I_k$, and after the row operation the P, T-entry is

$$y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} I_k - (-1)^k \sum_{j=1}^{k} (-1)^j \Big(\prod_{i=1}^{k} y_i^{\phi_i(j)} \Big) (R_{P_{(j)}})_T.$$

There is only one non-zero term in this sum. The only way the elements of T, which were already distributed in the blocks of π can still be distributed when we switch to the new set of dividers $P_{(j)}$ is if we remove the last divider p_k, thus merging π_k and the "back half" of π_1.

So the P, T-entry is actually

$$\begin{aligned} & y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} I_k - (-1)^k (-1)^k \Big(\prod_{i=1}^{k} y_i^{\phi_i(k)} \Big) (R_{P_{(k)}})_T \\ = \; & y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} I_k - y_1^{n+1-p_k} y_1^{t_1-1} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} I_k \\ = \; & \big(y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} - y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} \big) I_k \\ = \; & 0. \end{aligned}$$

Case 2.2: $t_i \in \pi_{i+1}, 1 \leq i \leq k-1, t_k \in \pi_1$

In particular, $t_k \in \{p_k, \ldots, n\}$, the back half of π_1. Then

$$(\Pi_{n,k}(\mathbf{y}))_{P,T} = y_1^{t_k-p_k} y_2^{t_1-p_1} \cdots y_k^{t_{k-1}-p_{k-1}} \alpha,$$

and after the row operation the P, T-entry is

$$y_1^{t_k-p_k} y_2^{t_1-p_1} \cdots y_k^{t_{k-1}-p_{k-1}} \alpha - (-1)^k \sum_{j=1}^{k} (-1)^j \Big(\prod_{i=1}^{k} y_i^{\phi_i(j)} \Big) (R_{P_{(j)}})_T.$$

As in Case 2.1, there is only one non-zero term in the sum. This time the only way the elements of T can still be distributed when we switch to the new set of dividers $P_{(j)}$ is if we remove the first divider p_1, thus merging the "front half" of π_1 and π_2.

So the P, T-entry is actually

$$
\begin{aligned}
& y_1^{t_k-p_k} y_2^{t_1-p_1} \cdots y_k^{t_{k-1}-p_{k-1}} \alpha - (-1)^k (-1)^1 \Big(\prod_{i=1}^{k} y_i^{\phi_i(1)} \Big) (R_{P_{(1)}})_T \\
= \; & y_1^{t_k-p_k} y_2^{t_1-p_1} \cdots y_k^{t_{k-1}-p_{k-1}} \alpha + (-1)^k y_1^{n+1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}} y_1^{t_1-1} y_2^{t_2-p_2} \cdots y_k^{t_k-p_k} I_k \\
= \; & y_1^{t_k-p_k} y_2^{t_1-p_1} \cdots y_k^{t_{k-1}-p_{k-1}} \alpha + (-1)^k y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} I_k \\
= \; & y_1^{t_k-p_k} y_2^{t_1-p_1} \cdots y_k^{t_{k-1}-p_{k-1}} \big(\alpha + (-1)^k y_1^{n+t_1-t_k} y_2^{t_2-t_1} \cdots y_k^{t_k-t_{k-1}} I_k \big).
\end{aligned}
$$

Note that in this case $T \geq P$ since $t_1 \in \pi_2$ implies $t_1 \geq p_1$. If $T = P$ we have

$$
\alpha + (-1)^k y_1^{n+t_1-t_k} y_2^{t_2-t_1} \cdots y_k^{t_k-t_{k-1}} I_k.
$$

Case 2.3: T is not distributed in the blocks of π.
In this case, $(\Pi_{n,k}(\mathbf{y}))_{P,T} = 0$, so after the row operation the P, T-entry is

$$
0 - (-1)^k \sum_{j=1}^{k} (-1)^j \Big(\prod_{i=1}^{k} y_i^{\phi_i(j)-1} \Big) (R_{P_{(j)}})_T.
$$

The only way any of the terms in the sum can be non-zero is if

1. π_1 contains two elements of T, one in the front half and the other in the back half,

2. there exists some $m, 2 \leq m \leq k$, such that π_m contains no elements of T, and

3. π_i contains exactly one element of T for all other $i, 2 \leq i \leq k$.

If 1, 2, and 3 all hold, then exactly two terms in the sum will be non-zero. Either we remove the divider p_{m-1}, merging blocks π_m and π_{m-1}, or we remove the divider p_m, merging blocks π_m and π_{m+1}.

So the (P, T)-entry becomes

$$
\begin{aligned}
& 0 - (-1)^k \Big((-1)^{m-1} y_m^{p_m-p_{m-1}} \cdots y_k^{p_k-p_{k-1}} y_1^{n+1-p_k} (R_{P_{(m-1)}})_T \\
& \qquad + (-1)^m y_{m+1}^{p_{m+1}-p_m} \cdots y_k^{p_k-p_{k-1}} y_1^{n+1-p_k} (R_{P_{(m)}})_T \Big) \\
= \; & (-1)^{k+1} \Big((-1)^{m-1} y_m^{p_m-p_{m-1}} \cdots y_k^{p_k-p_{k-1}} y_1^{n+1-p_k} y_1^{t_1-1} y_2^{t_2-p_1} \cdots y_{m-1}^{t_{m-1}-p_{m-2}} y_m^{t_m-p_m} \cdots y_k^{t_k-p_k} I_k \\
& \qquad + (-1)^m y_{m+1}^{p_{m+1}-p_m} \cdots y_k^{p_k-p_{k-1}} y_1^{n+1-p_k} y_1^{t_1-1} y_2^{t_2-p_1} \cdots y_m^{t_m-p_{m-1}} y_{m+1}^{t_{m+1}-p_{m+1}} \cdots y_k^{t_k-p_k} I_k \Big) \\
= \; & (-1)^{k+m} \Big(y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} - y_1^{n+t_1-p_k} y_2^{t_2-p_1} \cdots y_k^{t_k-p_{k-1}} \Big) I_k \\
= \; & 0.
\end{aligned}
$$

So after the row operations the matrix is upper triangular, with I_k's down the first $\binom{n-1}{k-1}$ diagonal entries, corresponding to the divider sets $P = \{p_1, \ldots, p_k\}$ which contain

1. The remaining $\binom{n-1}{k}$ diagonal entries are of the form $\alpha+(-1)^k y_1^{n+p_1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}} I_k$, where $P \in \binom{[n]\setminus 1}{k}$. Thus

$$\begin{aligned}
\det(\Pi^c_{n,k}(\mathbf{y})) &= \prod_{P\in\binom{[n]\setminus 1}{k}} \det\left(\alpha + (-1)^k y_1^{n+p_1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}} I_k\right) \\
&= \prod_{P\in\binom{[n]\setminus 1}{k}} \left((-1)^{k-1} + (-1)^{k^2}\left(y_1^{n+p_1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}}\right)^k\right) \\
&= \prod_{P\in\binom{[n]\setminus 1}{k}} (-1)^k \left(\left(y_1^{n+p_1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}}\right)^k - 1\right) \\
&= (-1)^{k\binom{n-1}{k}} \prod_{P\in\binom{[n]\setminus 1}{k}} \left(\left(y_1^{n+p_1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}}\right)^k - 1\right).
\end{aligned}$$

□

Corollary 4.6.7. *$A^c_{n,k}(\mathbf{y})$ is semisimple for generic $\mathbf{y} = (y_1, \ldots, y_k)$, in particular for all $\mathbf{y} \in \mathbb{C}^k$ such that $(y_1^{n+p_1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}})^k \neq 1$ for any $P = \{p_1, \ldots, p_k\} \subseteq [n]$.*

Proof. For any $a \in \mathbb{C}$ such that $a^k \neq 1$, the element $\alpha - aI_k$ is invertible in $\mathbb{C}S_k$. Indeed, $(\alpha-aI_k)\sum_{i=0}^{k-1} a^{k-1-i}\alpha^i = (1-a^k)I_k$. So if $\mathbf{y} \in \mathbb{C}^k$ satisfies $(y_1^{n+p_1-p_k} y_2^{p_2-p_1} \cdots y_k^{p_k-p_{k-1}})^k \neq 1$ for any $P = \{p_1, \ldots, p_k\} \subseteq [n]$, then the determinant of the sandwich matrix $\Pi^c_{n,k}(\mathbf{y})$ is the product of invertible elements of $\mathbb{C}S_k$ and hence is itself invertible. Thus we can compute the inverse of $\Pi^c_{n,k}(\mathbf{y})$ via cofactors. Then, by McAlister's result, $A^c_{n,k}(\mathbf{y})$ is semisimple. □

Chapter 5

The Radical

In this chapter we collect together several results concerning the radical $\sqrt{A_{n,k}}$ of the undeformed algebra $A_{n,k}$. We first characterize the radical in the special case $k = 1$, then give a more general eigenvalue result for $M_{n,k}$, the Gram matrix whose nullspace is the radical. We verify that the nullspace characterization of $\sqrt{A_{n,k}}$ coincides with another characterization in terms of the sandwich matrix, and exhibit several classes of elements in the radical. Finally, we describe two interesting actions of S_n on $\sqrt{A_{n,k}}$, one by conjugation and the other by left multiplication. We give a character result for the case $k = 1$.

5.1 The radical in the case $k = 1$

There are two cases in which it is possible to easily describe the radical of $A_{n,k}$. First of all, $A_{n,n}$ is isomorphic to the group algebra of the Symmetric Group, and thus is semisimple and has a trivial radical. At the other end of the spectrum is $A_{n,1}$, which is almost nilpotent. As the following result shows, its semisimple part is one-dimensional.

Proposition 5.1.1. *$\sqrt{A_{n,1}}$ is the ideal of $A_{n,1}$ consisting of all linear combinations of maps whose coefficients sum to zero. The dimension of $\sqrt{A_{n,1}}$ is $n-1$, and $A_{n,1}/\sqrt{A_{n,1}} \cong \mathbb{C}$.*

Proof. Let f_i denote the constant map sending every element of $[n]$ to i. First of all, $A_{n,1}$ is not itself nilpotent, since $f_i^k = f_i$ for all $k = 1, 2, \ldots$. Thus $\dim\left(\sqrt{A_{n,1}}\right) \leq n-1$. On the other hand, if I is the subalgebra of $A_{n,1}$ consisting of all linear combinations of maps whose coefficients sum to zero, then $\dim(I) = n-1$, since I is spanned by the $n-1$ linearly independent elements $f_i - f_{i+1}, 1 \leq i \leq n-1$. Thus it remains to show that I is a nilpotent ideal.

Suppose $x = \sum\limits_{i=1}^{n} c_i f_i$, where $\sum\limits_{i=1}^{n} c_i = 0$. Then

$$\begin{aligned} xf_j &= \left(\sum_{i=1}^{n} c_i f_i\right) f_j \\ &= \sum_{i=1}^{n} c_i f_i \end{aligned}$$

and so the sum of the coefficients of xf_j is $\sum\limits_{i=1}^{n} c_i = 0$, and $xf_j \in I$. On the other hand,

$$\begin{aligned} f_j x &= f_j \left(\sum_{i=1}^{n} c_i f_i\right) \\ &= \sum_{i=1}^{n} c_i f_j f_i \\ &= \sum_{i=1}^{n} c_i f_j \\ &= \left(\sum_{i=1}^{n} c_i\right) f_j \\ &= 0. \end{aligned}$$

So by linearity, if $y = \sum\limits_{i=1}^{n} d_i f_i$ is an arbitrary element of $A_{n,1}$, we have $xy, yx \in I$. Thus I is an ideal of $A_{n,1}$. To show that I is a *nilpotent* ideal, let $x = \sum\limits_{i=1}^{n} c_i f_i \in I$ and $y = \sum\limits_{j=1}^{n} d_j f_j \in I$. Then

$$\begin{aligned} xy &= \left(\sum_{i=1}^{n} c_i f_i\right)\left(\sum_{j=1}^{n} d_j f_j\right) \\ &= \left(\sum_{i=1}^{n} c_i f_i \left(\sum_{j=1}^{n} d_j f_j\right)\right) \\ &= 0 \end{aligned}$$

by the calculation above. So $I^2 = 0$, and hence I is nilpotent. $\square$

5.2 The Gram matrix, and a maximum eigenvalue

Recall the following fact from Chapter 2, which states that the radical of a matrix algebra is the nullspace of its Gram matrix.

Fact 5.2.1. *Let A be a finite-dimensional associative algebra and $\{v_1, \ldots, v_n\}$ a basis of A. Define the $n \times n$ Gram matrix M for A by $(M)_{i,j} = tr(v_i v_j)$. Then A is semisimple if and only if $det(M) \neq 0$, and in fact $\sqrt{A} = null(A)$.*

Let $M_{n,k}$ be the Gram matrix associated to the algebra $A_{n,k}$. In Chapter 2 we gave the following formula for $M_{n,k}$.

Proposition 5.2.2.

$$(M_{n,k})_{i,j} = \begin{cases} S(n,k)k! & \text{if } w_i w_j \text{ has } k \\ & \text{fixed points, and} \\ 0 & \text{otherwise.} \end{cases}$$

As before we normalize $M_{n,k}$ by dividing by $S(n,k)k!$, and we consider the eigenvalues of the normalized matrix.

n	k	eigenvalues
2	2	1^2
2	1	$0^1, 2^1$
3	3	$-1^1, 1^5$
3	2	$-3^1, -2^2, -1^1, 0^5, 1^3, 2^2, 3^3, 4^1$
3	1	$0^2, 3^1$
4	4	$-1^7, 1^{17}$
4	3	$-5^1, -4^3, -\sqrt{15}^6, -\sqrt{12}^3, -3^1, -\sqrt{5}^9, -2^3,$
		$-1^3, 0^{55}, 1^6, 2^6, \sqrt{5}^9, 3^3, 4^6, \sqrt{12}^3, \sqrt{15}^6, 5^6, 6^1$
4	2	$-8^3, -\sqrt{28}^5, -2^{10}, 0^{39}, 2^{15}, \sqrt{28}^5, 8^6, 14^1$
4	1	$0^3, 4^1$

Notice that in the table the largest eigenvalue for $k < n$ always appears with multiplicity 1. This is not a coincidence, as the result below shows.

For a set partition π of $[n]$ let $[[\pi]]$ denote the product of the block sizes of π. For example, if $\pi = 13|2|456$, then $[[\pi]] = 2 \cdot 1 \cdot 3 = 6$.

Theorem 5.2.3. *$\lambda = \sum_\pi [[\pi]]$, where the sum is over all $\pi \in S([n], k)$ such that the elements $1, 2, \ldots, k$ lie in different blocks, is the unique maximum eigenvalue for the normalized Gram matrix $M_{n,k}$. The corresponding eigenvector is $\mathbf{v} = \sum_{w \in T_{n,k}} [[\pi(w)]]w$.*

Proof. The proof of this result is rather technical, so we begin by sketching the main ideas. It is not hard to verify that $\mathbf{v}$ is an eigenvector for $M_{n,k}$, with eigenvalue λ. To show that λ is the *unique maximum* eigenvalue takes a bit more work. We show that $M_{n,k}$ is irreducible, by first relating it to the adjacency matrix of a graph $\Gamma_{n,k}$, and then showing that $\Gamma_{n,k}$ is connected. The proof of connectedness is by far the most technical part of the argument. We use induction to pare the problem down to a single special case, then display an algorithm that finishes the job. Once we have shown that $M_{n,k}$ is irreducible, we apply the Perron-Frobenius Theorem to conclude that $M_{n,k}$ has a unique maximum eigenvalue. Finally, it follows that λ is this same unique maximum eigenvalue because the corresponding eigenvector has all positive entries. □

Definition 5.2.4. *An $n \times n$ matrix A is* reducible *if it is similar via a permutation of rows and columns to a matrix of the form* $\begin{pmatrix} X & 0 \\ Y & Z \end{pmatrix}$, *where X and Z are non-trivial square matrices. A is* irreducible *if it is not reducible.*

Theorem 5.2.5. *(Perron-Frobenius) Let A be a non-negative irreducible matrix. Then A has a positive eigenvalue r that is a simple root of the characteristic polynomial. The modulus of any other eigenvalue of A does not exceed r. To the "maximal" eigenvalue r there corresponds an eigenvector with all positive entries.*

There is a way of interpreting $M_{n,k}$ as something like the adjacency matrix of a certain graph $\Gamma_{n,k}$ (with loops contributing 1 instead of 2). This graph turns out to be connected, and so its adjacency matrix is irreducible. $M_{n,k}$ must also be irreducible, because the question of irreducibility depends only on the support of a matrix, i.e., on the location of the non-zero entries.

In the following discussion we refer to maps $w \in T_{n,k}$ as "words".

Proposition 5.2.6. *Let $\Gamma_{n,k}$ be the graph whose vertices are the words $w \in T_{n,k}$ with edges $w_i - w_j$ if $w_i w_j$ has k fixed points. Then $\Gamma_{n,k}$ is connected for $1 \leq k \leq n-1$.*

Note that in $\Gamma_{n,k}$ we have $\deg(w_i)$ equal to the number of w_j such that $w_i w_j$ and $w_j w_i$ have k fixed points, which is $\prod_{m=1}^{k} \left|w_i^{-1}(s_m)\right| = [[\pi(w_i)]]$, since for each m, w_j must send s_m into $w_i^{-1}(s_m)$, for each $m = 1, 2, \ldots, k$.

Proof. (Method 1: "Lifting paths") We use induction on n. Given two words u and v in $\Gamma_{n,k}$ we first connect them respectively to words u' and v' of a special type. We then drop down to words u'' and v'' in $\Gamma_{n-1,k}$, connect u'' and v'' by a path, and finally "lift" the path back to $\Gamma_{n,k}$, thus connecting u and v.

To begin we observe that

$$\begin{aligned}
|V(\Gamma_{n,k})| &= |T_{n,k}| \\
&= \binom{n}{k} S(n,k)k! \\
&= \left(\binom{n-1}{k-1} + \binom{n-1}{k}\right)\left(S(n-1,k-1) + kS(n-1,k)\right)k! \\
&= k\binom{n-1}{k-1} S(n-1,k-1)(k-1)! + k\binom{n-1}{k} S(n-1,k)k! \\
&\quad + k\binom{n-1}{k-1} S(n-1,k)k! + k\binom{n-1}{k} S(n-1,k-1)(k-1)! \\
&= k\binom{n-1}{k-1} S(n-1,k-1)(k-1)! + k\binom{n-1}{k} S(n-1,k)k! \\
&\quad + \frac{k^2}{n-k}\binom{n-1}{k} S(n-1,k)k! + (n-k)\binom{n-1}{k-1} S(n-1,k-1)(k-1)!.
\end{aligned}$$

This expression suggests that the words in $T_{n,k}$ can be divided into four disjoint classes.

Type 1. $n \in \mathrm{Im}(w)$, n is in a block of size 1

Type 2. $n \notin \mathrm{Im}(w)$, n is in a block of size 2 or greater

Type 3. $n \in \mathrm{Im}(w)$, n is in a block of size 2 or greater

Type 4. $n \notin \mathrm{Im}(w)$, n is in a block of size 1

Claim: The number of words of each type is as follows.

Type 1. $k\binom{n-1}{k-1}S(n-1,k-1)(k-1)!$

Type 2. $k\binom{n-1}{k}S(n-1,k)k!$

Type 3. $\frac{k^2}{n-k}\binom{n-1}{k}S(n-1,k)k!$

Type 4. $(n-k)\binom{n-1}{k-1}S(n-1,k-1)(k-1)!$

Proof of claim:

- To get a word w' of Type 1, start with a word $w \in T_{n-1,k-1}$ (there are $\binom{n-1}{k-1}S(n-1,k-1)(k-1)!$ such words), and pick $w(n)$ from $\mathrm{Im}(w) \cup \{n\}$ (k choices for $w(n)$). If $w(n) \neq n$, say $w(n) = a$, then send all elements of $w^{-1}(a)$ to n instead. For example,

$$\begin{array}{lll}
1441 \longrightarrow & 14415 & (w(5) = 5, \text{ so no is switch needed}) \\
 & 54451 & (w(5) = 1, \text{ so we send } \{1,4\} \text{ to 5 instead of 1}) \\
 & 15514 & (w(5) = 4, \text{ so we send } \{2,3\} \text{ to 5 instead of 4}).
\end{array}$$

  We can go backwards by finding the instances of n. If n appears at the end of the word, just truncate. Otherwise, replace all occurrences of n by $w(n)$.

- To get a word w' of Type 2, start with a word $w \in T_{n-1,k}$ (there are $\binom{n-1}{k}S(n-1,k)k!$ such words), and pick $w(n)$ from $\mathrm{Im}(w)$ (k choices). For example,

$$\begin{array}{ccc} 1442 & \longrightarrow & 14421 \\ & & 14422 \\ & & 14424. \end{array}$$

  We can go backwards by truncating.

- To get a word w' of Type 3, start with a word $w \in T_{n-1,k}$ (there are $\binom{n-1}{k}S(n-1,k)k!$ such words), pick an element a of $\mathrm{Im}(w)$ (k choices for a), replace all instances of a by n, then pick $w(n)$ from $(\mathrm{Im}(w)\backslash a) \cup \{n\}$ (k choices). This results in an overcount by a factor of $n-k$, since in going backwards there are $n-k$ elements that might have been removed. For example,

$$\begin{array}{ccccc} 1442 & \longrightarrow & 54422 & 54424 & 54425 \\ & & 15521 & 15522 & 15525 \\ & & 14451 & 14454 & 14455. \end{array}$$

  Going backwards we replace the instances of n by each of the $n-k$ elements of $[n-1]\backslash\mathrm{Im}(w)$. For example,

$$\begin{array}{ccc} 14451 & \longrightarrow & 1442 \\ & & 1443. \end{array}$$

- To get a word w' of Type 4, start with a word $w \in T_{n-1,k-1}$ (there are $\binom{n-1}{k-1}S(n-1,k-1)(k-1)!$ such words), and pick $w(n)$ from $[n-1]\backslash\mathrm{Im}(w)$ ($n-k$ choices). For example,

$$\begin{array}{ccc} 1441 & \longrightarrow & 14412 \\ & & 14413. \end{array}$$

  As in Type 2 we can go backwards by truncating. This completes the proof of the claim.

As mentioned above, given two words $u, v \in \Gamma_{n,k}$, we shall drop down to $\Gamma_{n-1,k}$, connect the corresponding words there, then lift the path back.

Suppose $x = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_k \\ a_1 & a_2 & \dots & a_k \end{pmatrix}$ and $y = \begin{pmatrix} \nu_1 & \nu_2 & \dots & \nu_k \\ b_1 & b_2 & \dots & b_k \end{pmatrix}$ are connected by an edge in $\Gamma_{n-1,k}$. (This means that $b_j \in \pi_i$ if and only if $a_i \in \nu_j$.) We lift x to $\bar{x} = \begin{pmatrix} \pi_1 & \pi_2 & \dots & \pi_{k-1} & \pi'_k \\ a_1 & a_2 & \dots & a_{k-1} & a_k \end{pmatrix}$ and y to $\bar{y} = \begin{pmatrix} \nu_1 & \nu_2 & \dots & \nu_{k-1} & \nu'_k \\ b_1 & b_2 & \dots & b_{k-1} & b_k \end{pmatrix}$, where $\pi'_k = \pi_k \cup \{n\}$ and $\nu'_k = \nu_k \cup \{n\}$.

Notice that $\bar{x}$ and $\bar{y}$ are Type 2 words in $\Gamma_{n,k}$, and if $x - y$ in $\Gamma_{n-1,k}$ then $\bar{x} - \bar{y}$ in $\Gamma_{n,k}$. (n does not appear in the image of either $\bar{x}$ or $\bar{y}$, so $\bar{x}\bar{y}$ has k fixed points if and only if xy has k fixed points.)

For example, the edge $23113 - 41221$ in $\Gamma_{5,3}$ lifts to the edge $231133 - 412211$ in $\Gamma_{6,3}$.

As mentioned above, this lifting process always yields Type 2 words in $\Gamma_{n,k}$, i.e., words in which n is not in the image, and lies in a block of size at least 2. So the question becomes, given an arbitrary word can we connect it to a Type 2 word?

- Suppose w is Type 1, i.e., $w = \begin{pmatrix} \pi_1 & \dots & \pi_i & \dots & \pi_{k-1} & \{n\} \\ a_1 & \dots & n & \dots & a_{k-1} & a_k \end{pmatrix}$. Then construct $u = \begin{pmatrix} \nu_1 & \dots & \nu_r & \dots & \nu_s & \dots & \nu_k \\ b_1 & \dots & n & \dots & b_s & \dots & b_k \end{pmatrix}$ where $n \in \nu_s$ and $|\nu_s| \geq 2$ so that $a_k \in \nu_r, b_s \in \pi_i$, and in general $b_l \in \pi_m$ if and only if $a_m \in \nu_l$. Then u is Type 3 and $w - u$ in $\Gamma_{n,k}$.

  For example, $\begin{matrix} w = 25223 & - & u = 21522 \\ \text{Type 1} & & \text{Type 3} \end{matrix}$. Here we have complete freedom to choose the second and fourth entries of u (from among $\{1, 2, 5\}$), so we can ensure that 5 is in a block of size at least 2.

- Suppose w is Type 4, i.e., $w = \begin{pmatrix} \pi_1 & \dots & \pi_{k-1} & \{n\} \\ a_1 & \dots & a_{k-1} & a_k \end{pmatrix}, n \notin \text{Im}(w)$. Then construct $u = \begin{pmatrix} \nu_1 & \dots & \nu_r & \dots & \nu_s & \dots & \nu_k \\ b_1 & \dots & n & \dots & b_s & \dots & b_k \end{pmatrix}$, where $n \in \nu_s, |\nu_s| \geq 2, a_k \in \nu_r$, and in general $b_l \in \pi_m$ if and only if $a_m \in \nu_l$. Then u is Type 3 and $w - u$ in $\Gamma_{n,k}$.

  For example, $\begin{matrix} w = 24143 & - & u = 31522 \\ \text{Type 4} & & \text{Type 3} \end{matrix}$. Here we are free to choose the fifth entry of u, so we can ensure that n is in a block of size at least 2.

- Suppose w is Type 3, i.e., $w = \begin{pmatrix} \pi_1 & \dots & \pi_i & \dots & \pi_j & \dots & \pi_k \\ a_1 & \dots & n & \dots & a_j & \dots a_k & \end{pmatrix}$, where $n \in \pi_j$ and $|\pi_j| \geq 2$. Then construct $u = \begin{pmatrix} \nu_1 & \dots & \nu_l & \dots & \nu_k \\ b_1 & \dots & b_l & \dots & b_k \end{pmatrix}$, where $n \in \nu_l$ and $|\nu_l| \geq 2$ so that $b_q \in \pi_p$ if and only if $a_p \in t_q$, and $n \notin \text{Im}(u)$. Then u is Type 2 and $w - u$ in $\Gamma_{n,k}$.

  For example, $\begin{matrix} w = 21522 & - & u = 24333 \\ \text{Type 3} & & \text{Type 2} \end{matrix}$. Here we are free to choose the third and fourth entries of u, so we can ensure that n is in a block of size at least 2.

So, given any $u, v \in \Gamma_{n,k}$, we can first find $u', v' \in \Gamma_{n,k}$, both of Type 2, such that $u - \dots - u'$ and $v - \dots - v'$ in $\Gamma_{n,k}$. Then we can drop down to $u'', v'' \in \Gamma_{n-1,k}$ by simply removing n from its respective block in the two words. By induction, $\Gamma_{n-1,k}$ is connected, so we can find a path $u'' - \dots - v''$ in $\Gamma_{n-1,k}$, then lift the path to a path $u' - \dots - v'$ of Type 2 words in $\Gamma_{n,k}$. This produces a path $u - \dots - u' - \dots - v' - \dots - v$ in $\Gamma_{n,k}$ connecting u and v.

Example 5.2.7. *Suppose $u = 4141, v = 2223 \in T_{4,2}$. First we connect u to a Type 2 word u':*

$$\begin{matrix} u = 4141 & - & u' = 2111, \\ \textit{Type 3} & & \textit{Type 2} \end{matrix}$$

and v to a Type 2 word v':

$$\begin{matrix} v = 2223 & - & 2244 & - & v' = 2233. \\ \textit{Type 4} & & \textit{Type 3} & & \textit{Type 2} \end{matrix}$$

Then we drop down to $u'', v'' \in \Gamma_{3,2}$*:*

$$\begin{aligned} u' = 2111 &\longrightarrow u'' = 211, \\ v' = 2233 &\longrightarrow v'' = 233. \end{aligned}$$

In $\Gamma_{3,2}$ *we have the path* $u'' = 211 - 313 - 223 = v''$*. This path lifts to* $u' = 2111 - 3133 - 2233 = v''$*. So we have the path* $u = 4141 - 2111 - 3133 - 2233 - 2244 - 2223 = v$ *in* $\Gamma_{4,2}$*.*

At this point we encounter a serious problem, for the argument above only works if $k \leq n-2$. If $k = n-1$ we drop down to $\Gamma_{n-1,n-1}$ which is *not* connected. In general $\Gamma_{m,m}$ consists of isolated vertices (involutions) and dimers $w - w^{-1}$. So we need to show by some other means that $\Gamma_{n,n-1}$ is connected. This we shall do in the following lemma.

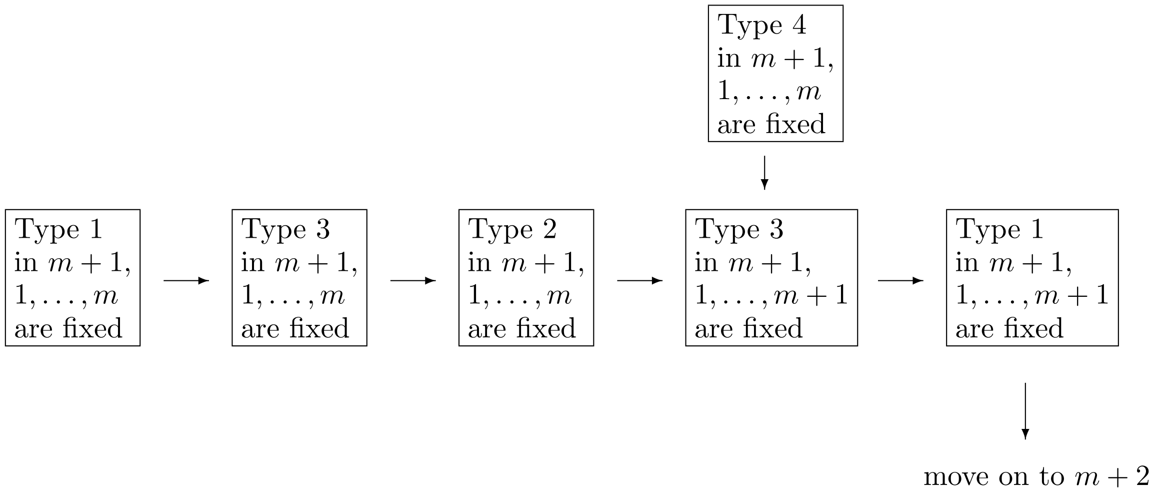
Lemma 5.2.8. *Each word of* $\Gamma_{n,n-1}$ *is connected by a path to the "smallest" word*

$$w_0 = \begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & \{n, n-1\} \\ 1 & 2 & 3 & \dots & n-2 & n-1 \end{pmatrix}.$$

Proof. We exhibit an algorithm to construct such a path. Say that a word is "Type i in j" if it satisfies the definition of Type i given above, with j playing the roll of n. For example, the word $u = 125546$ is Type 2 in 3, since $3 \notin \mathrm{Im}(u)$ and 3 is in a block of size at least 2 in $\pi(u)$. On the other hand, $v = 123466$ is Type 1 in 4, since $4 \in \mathrm{Im}(v)$ and 4 is in a block of size 1 in $\pi(v)$. In what follows we show how to connect an arbitrary word with an initial string of m fixed points, $0 \leq m \leq n-2$, to a word that is Type 1 in $m+1$ with an initial string of $m+1$ fixed points. By induction we can then connect any given word u to a word that is Type 1 in $n-1$ with an initial string of $n-1$ fixed points, i.e., a word of the form $w = \begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 1 & 2 & 3 & \dots & n-2 & n-1 & a \end{pmatrix}$, where $a \in [n-2]$. But then we are done, for w is clearly adjacent to w_0.

Since we are dealing with words in $\Gamma_{n,n-1}$, every word has partition consisting of a single block of size 2, and $n-2$ singleton blocks. Thus, given $u \in \Gamma_{n,n-1}$, if we want to find a v such that $u - v$, i.e., such that uv has $n-1$ fixed points, we have little choice. Each element $a \in \mathrm{Im}(u)$ must be sent by v to $u^{-1}(a)$, which has size 1 in all but one case. In that one case we have just two choices. We are also free to choose $v(b)$, where b is the one element of $[n]$ not appearing in $\mathrm{Im}(u)$. For example, if $u = 422516$ then $v = 5xy146$, where x must be either 2 or 3, and y is almost arbitrary, the one restriction being that $y \in \{1, 4, 5, 6, x\}$.

We now describe how to connect a given word w with an initial string of m fixed points to a word w' that is in some way "closer" to having an initial string of $m+1$ fixed points. The argument is rather technical, and it may pay to keep the following diagram in mind.



We shall see that wherever we are in the diagram, we can follow the arrows to increase the length of the initial string of fixed points.

- Suppose an arbitrary word w is Type 1 in $m+1$ and has an initial string of m fixed points, where $0 \leq m \leq n-2$. This means $m+1 \in \text{Im}(w)$, and $m+1$ is in a block of size 1. In constructing w' so that $w - w'$ we can force $m+1$ to be in a block of size 2 by setting $w'(b) = w'(m+1)$, where as above b is the one element of $[n]$ not appearing in $\text{Im}(w)$. If $a \in \{1,\dots,m\}$ is in a block of size 2 in w, namely $\{a, a'\}$, where $a' > m$, choose $w'(a) = a$, rather than a'. This ensures that w' will also have an initial string of m fixed points. Finally note we must have $m+1 \in \text{Im}(w')$ since $m+1$ was in a block of size 1 in w. So w' is Type 3 in $m+1$, with an initial string of m fixed points.

- Suppose now that w is Type 3 in $m+1$ and has an initial string of m fixed points. This means $m+1 \in \text{Im}(w)$, and $m+1$ is in a block of size 2. So we can construct w' with $w - w'$ such that $m+1 \notin \text{Im}(w')$. We can also make sure that $m+1$ is in a block of size 2 in w' by setting $w'(b) = w'(m+1)$, where $b \notin \text{Im}(w)$. As above the first m elements remain fixed, and so w' is Type 2 in $m+1$, with an initial string of m fixed points.

- Suppose now that w is Type 2 in $m+1$ and has an initial string of m fixed points. This means $m+1 \notin \text{Im}(w)$, and $m+1$ is in a block of size 2. Since $m+1$ is in a block of size 2 of w we can choose $m+1$ to be in the image of w'. Thus when we go to pick $w'(m+1)$ "almost arbitrarily", we can pick it to be $m+1$. The first m elements are still fixed by w', so now w' has an initial string of $m+1$ fixed points. $m+1 \in \text{Im}(w')$, and $m+1$ is in a block of size 2, so w' is Type 3 in $m+1$, with an initial string of $m+1$ fixed points.

- Suppose instead that w is Type 4 in $m+1$ and has an initial string of m fixed points. This means $m+1 \notin \text{Im}(w)$, and $m+1$ is in a block of size 1. Then we are forced to have $m+1 \in \text{Im}(w')$, and thus we can choose $w'(m+1)$ (the element of w' with the most freedom) to be $m+1$ as well. The first m elements are still fixed by w', so now w' has an initial string of $m+1$ fixed points. $m+1 \in \text{Im}(w')$, and $m+1$ is in a block of size 2, so w' is Type 3 in $m+1$, with an initial string of $m+1$ fixed points.

- Suppose finally that w is Type 3 in $m+1$ and has an initial string of $m+1$ fixed points. This means $m+1 \in \text{Im}(w)$, and $m+1$ is in a block of size 2. In constructing w' with an initial string of $m+1$ fixed points and $w - w'$ pick $w'(m+1) = m+1$. No other element is forced to be sent to $m+1$, so we can ensure that $m+1$ is in a block of size 1 in w'. $m+1 \in \text{Im}(w')$, and $m+1$ is in a block of size 1, so w' is Type 1 in $m+1$, with an initial string of $m+1$ fixed points.

So we have connected w to a new word w' of Type 1 in $m+1$ with an initial string of $m+1$ fixed points. Now we can continue the process, for w' is of some type in $m+2$, and we repeat the necessary steps above to connect w' to a word w'' with an initial string of $m+2$ fixed points.

Eventually we arrive at a word w of Type 1 in $n-1$, with an initial string of $n-1$ fixed points. Being Type 1 in $n-1$ means that $n-1 \in \text{Im}(w)$ and $n-1$ is in a block of size 1. So $w = \begin{pmatrix} 1 & 2 & 3 & \dots & n-2 & n-1 & n \\ 1 & 2 & 3 & \dots & n-2 & n-1 & a \end{pmatrix}$, where $a \in [n-2]$. But then $ww_0 = w_0$ has $n-1$ fixed points, and so w is adjacent to w_0 and we are done. □

Example 5.2.9. *Let $n = 6$ and suppose $w = 422516$. Our goal is to connect w to $w_0 = 123455$. To start, notice that w is Type 1 in 1, with no initial string of fixed points.*

422516	–	525146	*Type 3 in 1*
	–	424536	*Type 2 in 1*
	–	125146	*Type 3 in 1, 1 is fixed*
	–	122536	*Type 1 in 1, 1 is fixed*
	=	122536	*Type 3 in 2, 1, 2 are fixed*
	–	125546	*Type 1 in 2, 1, 2 are fixed*
	=	125546	*Type 2 in 3, 1, 2 are fixed*
	–	123536	*Type 3 in 3, 1, 2, 3 are fixed*
	–	123446	*Type 1 in 3, 1, 2, 3 are fixed*
	=	123446	*Type 3 in 4, 1, 2, 3, 4 are fixed*
	–	123466	*Type 1 in 4, 1, 2, 3, 4 are fixed*
	=	123466	*Type 2 in 5, 1, 2, 3, 4 are fixed*
	–	123455	

(Method 2: "Algorithmic") The algorithm described above actually works for any n and $k, k < n$. Let $w_0 = \begin{pmatrix} 1 & 2 & 3 & \dots & k-1 & \{k, \dots, n\} \\ 1 & 2 & 3 & \dots & k-1 & k \end{pmatrix}$. The algorithm allows us to connect an arbitrary $w \in T_{n,k}$ to w_0, in fact the only difference is that we have more choices in constructing the w'.

Example 5.2.10. *Let* $n = 8$ *and* $k = 5$ *and suppose* $w = 46221332$. *Our goal is to connect* w *to* $w_0 = 12345555$. *To start, notice that* w *is Type 1 in 1, with no initial string of fixed points.*

46221332	–	53615255	*Type 3 in 1*
	–	46245366	*Type 2 in 1*
	–	13615222	*Type 3 in 1, 1 is fixed*
	–	16265366	*Type 1 in 1, 1 is fixed*
	=	16265366	*Type 3 in 2, 1 is fixed*
	–	13635444	*Type 2 in 2, 1 is fixed*
	–	12265366	*Type 3 in 2, 1, 2 are fixed*
	–	12665444	*Type 1 in 2, 1, 2 are fixed*
	=	12665444	*Type 2 in 3, 1, 2 are fixed*
	–	12365366	*Type 3 in 3, 1, 2, 3 are fixed*
	–	12345444	*Type 1 in 3, 1, 2, 3 are fixed*
	=	12345444	*Type 3 in 4, 1, 2, 3, 4 are fixed*
	–	12345555	

Now it is left to prove the theorem. Recall that $\mathbf{v}$ was defined by

$$\mathbf{v} = \sum_{w \in T_{n,k}} [[\pi(w)]]w.$$

. So

$$\begin{aligned}
(M_{n,k}\mathbf{v})_i &= \sum_{j=1}^{\binom{n}{k}S(n,k)k!} (M_{n,k})_{i,j}\mathbf{v}_j \\
&= \sum_{\substack{w_j \text{ s.t. } w_i w_j \\ \text{has } k \text{ fixed points}}} [[\pi(w_j)]] \\
&= \sum_{\substack{\text{partitions } \pi \text{ of } [n] \\ \text{into } k \text{ blocks s.t.} \\ \text{each element of } \mathrm{Im}(w_i) \\ \text{lies in a different block}}} [[\pi]] \cdot \# \left\{ \begin{array}{c} w_j \text{ s.t. } w_i w_j \text{ has } k \text{ fixed points} \\ \text{and } \pi(w_j) = \pi \end{array} \right\} \\
&= \sum_{\substack{\text{partitions } \pi \text{ of } [n] \\ \text{into } k \text{ blocks s.t.} \\ \text{each element of } \mathrm{Im}(w_i) \\ \text{lies in a different block}}} [[\pi]][[\pi(w_i)]] \\
&= \left(\sum_{\substack{\text{partitions } \pi \text{ of } [n] \\ \text{into } k \text{ blocks s.t.} \\ 1, 2, \ldots, k \\ \text{lie in a different blocks}}} [[\pi]] \right) [[\pi(w_i)]] \\
&= \lambda \mathbf{v}_i
\end{aligned}$$

So $\mathbf{v}$ is an eigenvector of $M_{n,k}$, and since $\mathbf{v}$ has all positive entries, it must correspond to the unique maximum eigenvalue. $\square$

Note that we can write the expression for λ in several ways.

$$\begin{aligned}
\lambda &= \sum_{\substack{\text{partitions } \pi \text{ of } [n] \\ \text{into } k \text{ blocks s.t.} \\ 1,2,\ldots,k \\ \text{lie in different blocks}}} [[\pi]] \\
&= \sum_{\substack{\text{compositions} \\ p \vDash n \text{ with} \\ k \text{ nonzero parts}}} \binom{n-k}{p_1-1, p_2-1, \ldots, p_k-1} p_1 p_2 \cdots p_k \\
&= \sum_{\substack{\lambda \vdash n \\ \lambda \text{ has } k \text{ parts}}} \binom{k}{m_1, m_2, \ldots, m_r} \binom{n-k}{\lambda_1-1, \lambda_2-1, \ldots, \lambda_k-1} \lambda_1 \lambda_2 \cdots \lambda_k,
\end{aligned}$$

where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ has multiplicities $m_1, m_2, \ldots, m_r$.

Making use of the multinomial theorem allows us to rewrite λ in an even simpler form, shown below.

Corollary 5.2.11. *The unique maximum eigenvalue of the normalized Gram matrix $M_{n,k}$ is*

$$\lambda = \sum_{j=0}^{k} \binom{n-k}{j} \binom{k}{j} j! k^{n-k-j}.$$

Proof. By the Multinomial Theorem we have

$$\begin{aligned}
(x_1 x_2 \cdots x_k)(x_1 + x_2 + \ldots + x_k)^{(n-k)} &= (x_1 x_2 \cdots x_k) \sum_{\substack{p \vDash n, \\ p_i \geq 0}} \binom{n-k}{p_1, p_2, \ldots, p_k} x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k} \\
&= \sum_{\substack{p \vDash n, \\ p_i \geq 0}} \binom{n-k}{p_1, p_2, \ldots, p_k} x_1^{p_1+1} x_2^{p_2+1} \cdots x_k^{p_k+1} \\
&= \sum_{\substack{p \vDash n, \\ p_i \geq 1}} \binom{n-k}{p_1-1, p_2-1, \ldots, p_k-1} x_1^{p_1} x_2^{p_2} \cdots x_k^{p_k}.
\end{aligned}$$

Differentiating both sides once with respect to each variable gives

$$\sum_{S \subseteq [k]} (n-k)_{|S|} (x_1 + x_2 + \ldots + x_k)^{n-k-|S|} \prod_{i \in S} x_i$$

on the left-hand side, and

$$\sum_{\substack{p \vDash n,\\ p_i \geq 1}} \binom{n-k}{p_1-1, p_2-1, \ldots, p_k-1} p_1 p_2 \cdots p_k x_1^{p_1-1} x_2^{p_2-1} \cdots x_k^{p_k-1} \tag{5.1}$$

on the right-hand side. Evaluating both sides at $x_1 = x_2 = \ldots = x_k = 1$ gives

$$\begin{aligned} \sum_{\substack{p \vDash n,\\ p_i \geq 1}} \binom{n-k}{p_1-1, p_2-1, \ldots, p_k-1} p_1 p_2 \cdots p_k &= \sum_{S \subseteq [k]} (n-k)_{|S|} k^{n-k-|S|} \\ &= \sum_{j=0}^{k} \binom{n-k}{j} \binom{k}{j} j! k^{n-k-j}. \end{aligned}$$

□

5.3 Equations for the radical

The following result of McAlister [9] gives a characterization of the radical of a Munn matrix algebra.

Proposition 5.3.1. *(McAlister) Let $\mathcal{A} = \mathcal{M}(A; m, n; \Pi)$ be a Munn matrix algebra. Then*

$$\sqrt{\mathcal{A}} = \{x \in \mathcal{A} | \Pi x \Pi \in (\sqrt{A})_{m,n}\}.$$

So in particular, if A is semisimple, as in our case,

$$\sqrt{\mathcal{A}} = \{x \in \mathcal{A} | \Pi x \Pi = 0\}.$$

Recall that we also have a characterization of $\sqrt{\mathcal{A}}$ as the nullspace of the corresponding Gram matrix. A natural question to ask is whether these two characterizations are related. The following result states that in the case of $A_{n,k}$ they actually give the same equations.

Proposition 5.3.2. *In the case of $A_{n,k}$ the condition $\Pi x \Pi = 0$ gives the same equations on the entries of x as does $x \in \mathrm{null}(M_{n,k})$, where $M_{n,k}$ is the Gram matrix for $A_{n,k}$.*

Proof. We have seen that

$$(M_{n,k})_{u,v} = \begin{cases} S(n,k)k! & \text{if } uv \text{ has } k \text{ fixed points, and} \\ 0 & \text{otherwise.} \end{cases}$$

So given

$$x = \sum_{\substack{v \in T_{n,k} \\ uv \text{ has } k \text{ fixed points}}} c_v v \in A_{n,k},$$

the condition $M_{n,k}x = 0$ gives the set of equations $\left\{ \sum_{\substack{v \in T_{n,k} \\ uv \text{ has } k \text{ fixed points}}} c_v = 0 \right\}_{u \in T_{n,k}}$.

Now let $u = w_{\pi,S,\sigma}$. What is the coefficient of σ in $(\Pi x \Pi)_{\pi,S}$? First of all,

$$(\Pi x \Pi)_{\pi,S} = \sum_{\substack{\nu \in S([n],k) \\ T \in \binom{[n]}{k}}} (\Pi)_{\pi,T}(x)_{T,\nu}(\Pi)_{\nu,S}.$$

Recall that

$$(\Pi)_{\pi,S} = \begin{cases} \tau & \text{if } s_i \in \pi_{\tau(i)}, 1 \le i \le k, \text{ and} \\ 0 & \text{otherwise.} \end{cases}$$

In this situation, we say that S is *distributed in* π *by* τ. So

$$\begin{aligned} (\Pi x \Pi)_{\pi,S} &= \sum_{\alpha,\beta \in S_k} \sum_{\substack{\nu \in S([n],k),\, S \text{ distributed in } \nu \text{ by } \alpha \\ T \in \binom{[n]}{k},\, T \text{ distributed in } \pi \text{ by } \beta}} \beta(x)_{T,\nu}\alpha \\ &= \sum_{\gamma \in S_k} \sum_{\alpha,\beta \in S_k} \sum_{\substack{\nu \in S([n],k),\, S \text{ distributed in } \nu \text{ by } \alpha \\ T \in \binom{[n]}{k},\, T \text{ distributed in } \pi \text{ by } \beta}} c_{w_{\nu,T,\gamma}} \beta\gamma\alpha. \end{aligned}$$

Thus the coefficient of σ in $(\Pi x \Pi)_{\pi,S}$ is

$$\begin{aligned} &\sum_{\substack{\alpha,\beta,\gamma \in S_k \\ \beta\gamma\alpha = \sigma}} \sum_{\substack{\nu \in S([n],k),\, S \text{ distributed in } \nu \text{ by } \alpha \\ T \in \binom{[n]}{k},\, T \text{ distributed in } \pi \text{ by } \beta}} c_{w_{\nu,T,\gamma}} \\ = &\sum_{\substack{v \in T_{n,k} \\ uv \text{ induces } \sigma^2 \text{ on } S = \mathrm{Im}(u)}} c_v, \end{aligned}$$

since if S is distributed in ν by α and T in π by β we have

$$s_i \longrightarrow \nu_{\alpha(i)} \longrightarrow T_{\gamma\alpha(i)} \longrightarrow \pi_{\beta\gamma\alpha(i)} = \pi_{\sigma(i)} \longrightarrow s_{\sigma^2(i)}.$$

This argument works backwards, too. If uv induces σ^2 on $S = \mathrm{Im}(u)$, and $u = w_{\pi,S,\sigma}, v = w_{\nu,T,\gamma}$ there must exist $\alpha, \beta \in S_k$ such that S is distributed in ν by α and T in π by β as otherwise uv could not permute the elements of S, and hence the composition must satisfy $\sigma\beta\gamma\alpha = \sigma^2$, i.e., $\beta\gamma\alpha = \sigma$.

So $\Pi x \Pi = 0$ gives the set of equations $\left\{ \sum_{\substack{v \in T_{n,k} \\ uv \text{ induces } \sigma^2 \text{ on } S = \mathrm{Im}(u)}} c_v = 0 \right\}_{u \in T_{n,k}}$.

Now we claim that the two sets of equations are the same. In particular, if we define

$$\begin{aligned} A(u) &:= \{v \in T_{n,k} | uv \text{ induces } \sigma^2 \text{ on } S = \mathrm{Im}(u)\} \\ B(u) &:= \{v \in T_{n,k} | uv \text{ has } k \text{ fixed points}\}, \end{aligned}$$

and for $u = w_{\pi,S,\sigma}$ we define $u^* := w_{\pi,S,\sigma^{-1}}$, then $A(u) = B(u^*)$. Then since the correspondence $u \longrightarrow u^*$ is an involution, the two sets of equations are the same.

To prove that $A(u) = B(u^*)$, suppose that uv induces σ^2 on $S = \mathrm{Im}(u)$. Write $u = w_{\pi,S,\sigma}$ and $v = w_{\nu,T,\gamma}$. Since uv permutes the elements of S there must exists $\alpha, \beta \in S_k$ such that $s_i \in \nu_{\alpha(i)}, t_i \in \pi_{\beta(i)}, 1 \le i \le k$, and so

$$s_i \longrightarrow \nu_{\alpha(i)} \longrightarrow t_{\gamma\alpha(i)} \longrightarrow \pi_{\beta\gamma\alpha(i)} \longrightarrow s_{\sigma\beta\gamma\alpha(i)} = s_{\sigma^2(i)}.$$

Thus $\sigma\beta\gamma\alpha = \sigma^2$, i.e., $\beta\gamma\alpha = \sigma$. But then u^*v has k fixed points, since

$$s_i \longrightarrow \nu_{\alpha(i)} \longrightarrow t_{\gamma\alpha(i)} \longrightarrow \pi_{\beta\gamma\alpha(i)} = \pi_{\sigma(i)} \longrightarrow s_{\sigma^{-1}\sigma(i)} = s_i.$$

Similarly, if u^*v has k fixed points, then $(u^*)^*v = uv$ induces σ^2 on $S = \mathrm{Im}(u)$. So $A(u) = B(u^*)$ and we are done. ☐

Example 5.3.3. *Let $w = 3132$, so that $\pi(w) = 13|2|4, S(w) = \{1,2,3\}$, and $\phi(w) = (132)$. Then*

$$A(w) = \{w' \in T_{4,3} | ww' \text{ induces } \sigma^2 = (123) \text{ on } S = \{1,2,3\}\}.$$

If $(3132)w_{\nu,T,\gamma} = w_{\nu,S,\sigma^2}$, then $T = \mathrm{Im}(w')$ must be $\{1,2,4\}$ or $\{2,3,4\}$, and ν must be $1|2|34, 1|24|3$, or $14|2|3$. Here is a table.

ν	T	α	β	γ	$w' = w_{\nu,T,\gamma}$
$1\|2\|34$	$\{1,2,4\}$	*id*	*id*	(132)	4122
	$\{2,3,4\}$	*id*	(12)	(13)	4322
$1\|24\|3$	$\{1,2,4\}$	*id*	*id*	(132)	4121
	$\{2,3,4\}$	*id*	(12)	(13)	4323
$14\|2\|3$	$\{1,2,4\}$	*id*	*id*	(132)	4124
	$\{2,3,4\}$	*id*	(12)	(13)	4324

So

$$A(w) = \{4122, 4322, 4121, 4323, 4124, 4324\}.$$

Meanwhile, $w^* = w_{\pi,S,\sigma^{-1}} = 2321$, so

$$B(w^*) = \{w' \in T_{4,3} | w^*w' \text{ has three fixed points}\}$$

If $(2321)w'$ *fixes 1, 2, and 3, the first entry of* w' *must be 4, the second must be 1 or 3, the third must be 2, and the fourth can be any of the first three. So*

$$B(w^*) = \{4121, 4122, 4124, 4322, 4323, 4324\} = A(w).$$

5.4 Some elements of the radical

One of our original questions about the non-deformed algebras $A_{n,k}$ was whether we could provide some sort of combinatorial description for the radical $\sqrt{A_{n,k}}$, perhaps in the form of a spanning set or even a basis. As described in the propositions below, we have found several classes of elements in $\sqrt{A_{n,k}}$.

Proposition 5.4.1. *Fix a subset* $P = \{p_1, \ldots, p_k\} \subseteq [n]$ *and an integer partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ *of* n *into distinct parts,* $k > 1$*. Given a set partition* $\pi = \{\pi_1, \ldots, \pi_k\}$ *of type* λ *we associate a permutation* ν_π *by* $|\pi_i| = \lambda_{\nu_\pi(i)}$*. Then*

$$x_{\lambda,P} := \sum_{\pi} sgn(\nu_\pi) \sum_{\sigma \in S_k} sgn(\sigma) w_{\pi,P,\sigma} \in \sqrt{A_{n,k}},$$

where the outer sum is over all set partitions π *of type* λ*.*

Proof. For an *ordered* partition $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ of $[n]$ and a subset $P = \{p_1, p_2, \ldots, p_k\} \subseteq [n]$, let $w_{\pi,P} = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_k \\ p_1 & p_2 & \cdots & p_k \end{pmatrix}$, i.e., $w_{\pi,P}$ sends the elements of the block π_i to $p_i, 1 \leq i \leq k$. Every word has a unique representation in this form because the $p_1, p_2, \ldots, p_k$ are still taken in increasing order.

For example, if $\pi = (\{1,6\}, \{5\}, \{2,3,4\})$ and $P = \{2,3,4\}$ then

$$w_{\pi,P} = \begin{pmatrix} \{1,6\} & \{5\} & \{2,3,4\} \\ 2 & 4 & 5 \end{pmatrix} = 255542.$$

Note that $x_{\lambda,P}$ can be written using this new notation as

$$x_{\lambda,P} = \sum_{\nu \in S_k} \mathrm{sgn}(\nu) \sum_{\substack{\pi \text{ an ordered partition} \\ \text{of } [n] \text{ with } |\pi_i| = \lambda_{\nu(i)}}} w_{\pi,P}$$

Now let $w_{\sigma,S} = \begin{pmatrix} \sigma_1 & \sigma_2 & \cdots & \sigma_k \\ s_1 & s_2 & \cdots & s_k \end{pmatrix} \in T_{n,k}$. We claim that $x_{\lambda,P} w_{\sigma,S} = 0$. For a fixed $\pi, w_{\pi,P} w_{\sigma,S} = 0$ unless the S is distributed in the blocks of π, say $s_i \in \pi_{\tau(i)}$.

Thus

$$\begin{aligned}
x_{\lambda,P}w_{\sigma,S} &= \sum_{\nu\in S_k}\mathrm{sgn}(\nu)\sum_{\substack{\pi \text{ an ordered partition}\\ \text{of } [n] \text{ with } |\pi_i|=\lambda_{\nu(i)}}} w_{\pi,P}w_{\sigma,S}\\
&= \sum_{\nu\in S_k}\mathrm{sgn}(\nu)\sum_{\tau\in S_k}\sum_{\substack{\pi \text{ an ordered partition}\\ \text{of } [n] \text{ with } |\pi_i|=\lambda_{\nu(i)}\\ \text{and } s_i\in\pi_{\tau(i)}}} w_{\pi,P}w_{\sigma,S}\\
&= \sum_{\nu\in S_k}\mathrm{sgn}(\nu)\sum_{\tau\in S_k}\left(\begin{array}{c}\#\text{ ordered partitions } \pi \text{ with}\\ |\pi_i|=\lambda_{\nu(i)} \text{ and } s_i\in\pi_{\nu(i)}\end{array}\right)w_{\tau^{-1}(\sigma),P}\\
&= \sum_{\nu\in S_k}\mathrm{sgn}(\nu)\sum_{\tau\in S_k}\binom{n-k}{\lambda_{\nu(1)}-1,\lambda_{\nu(2)}-1,\ldots,\lambda_{\nu(k)}-1}w_{\tau^{-1}(\sigma),P}\\
&= \sum_{\nu\in S_k}\mathrm{sgn}(\nu)\sum_{\tau\in S_k}\binom{n-k}{\lambda_1-1,\lambda_2-1,\ldots,\lambda_k-1}w_{\tau^{-1}(\sigma),P}\\
&= \left(\sum_{\nu\in S_k}\mathrm{sgn}(\nu)\right)\left(\sum_{\tau\in S_k}\binom{n-k}{\lambda_1-1,\lambda_2-1,\ldots,\lambda_k-1}w_{\tau^{-1}(\sigma),P}\right)\\
&= 0\cdot\left(\sum_{\tau\in S_k}\binom{n-k}{\lambda_1-1,\lambda_2-1,\ldots,\lambda_k-1}w_{\tau^{-1}(\sigma),P}\right)\\
&= 0
\end{aligned}$$

□

Example 5.4.2. *For $n=3, k=2$ take $\lambda=21$. Then*

$$\begin{aligned}
x_{\lambda,\{1,2\}} &= 112+121+211-221-212-122,\\
x_{\lambda,\{1,3\}} &= 113+131+311-331-313-133, \text{ and}\\
x_{\lambda,\{2,3\}} &= 223+232+322-332-323-233.
\end{aligned}$$

Proposition 5.4.3. *Fix a subset $P=\{p_1,\ldots,p_{k+1}\}\subseteq[n]$ and a set partition $\pi=\{\pi_1,\ldots,\pi_k\}$ of $[n]$, $k<n$. Then*

$$x_{\pi,P} := \sum_{i=1}^{k+1}(-1)^i\sum_{\sigma\in S_k} sgn(\sigma)w_{\pi,P\setminus\{p_i\},\sigma}\in\sqrt{A_{n,k}}.$$

Proof. Given $w=\begin{pmatrix}\nu_1 & \nu_2 & \cdots & \nu_k\\ s_1 & s_2 & \cdots & s_k\end{pmatrix}\in T_{n,k}$ we need to show that $\mathrm{tr}(x_{\pi,P}w)=0$.

First of all, if the elements $s_1,\ldots,s_k$ are not in different blocks of π, then $w_{\pi,P\setminus p_i,\tau}w=0$ for all i and τ. Without loss of generality we can assume $s_1\in\pi_1,\ldots,s_k\in\pi_k$. If the

k elements of $P\backslash p_i$ are not in different blocks of ν, then $\mathrm{tr}(w_{\pi,P-p_i,\tau}w) = 0$. So

$$\mathrm{tr}(x_{\pi,P}w) = \mathrm{tr}\left(\sum_i (-1)^i \sum_{\tau\in S_k} \mathrm{sgn}(\tau) w_{\pi,P\backslash p_i,\tau} w\right),$$

where the sum is over all i such that the elements of $P\backslash p_i$ lie in different blocks of σ.

Now if the $(k+1)$-subset P does not have at least one element in each of the k blocks of ν, no such i exists and we are done. Otherwise we are in a situation where two elements of P, say p_i and p_j, are together in one block of ν while the rest are each in their own block. So

$$\mathrm{tr}(x_{\pi,P}w) = (-1)^i \mathrm{tr}\left(\sum_{\tau\in S_k} \mathrm{sgn}(\tau) w_{\pi,P\backslash p_i,\tau} w\right) + (-1)^j \mathrm{tr}\left(\sum_{\tau\in S_k} \mathrm{sgn}(\tau) w_{\pi,P\backslash p_j,\tau} w\right).$$

Which terms in these two sums survive?

Label $P\backslash p_i$ as $\{p'_1, p'_2, \ldots, p'_k\}$ so that $w_{\pi,P\backslash p_j,\tau} = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_k \\ p'_{\tau(1)} & p'_{\tau(2)} & \cdots & p'_{\tau(k)} \end{pmatrix}$. For $w_{\pi,P\backslash p_i,\tau}w$ to have k fixed points we need

$$p'_l \longleftarrow \pi_{\tau^{-1}(l)} \longleftarrow s_{\tau^{-1}(l)} \longleftarrow \nu_{\tau^{-1}(l)} \longleftarrow p'_l, \text{ for all } l = 1, 2, \ldots, k,$$

i.e., $p'_l \in \sigma_{\tau^{-1}(l)}$. This completely determines τ.

Let τ_i be the τ such that $p'_l \in \nu_{\tau_i^{-1}(l)}$ for $l = 1, \ldots, k$. Similarly, label the k elements of $P\backslash p_j$ as $\{p''_1, p''_2, \ldots, p''_k\}$ and let τ_j be the τ such that $p''_l \in \nu_{\tau_j^{-1}(l)}$ for $l = 1, \ldots, k$.

So

$$\begin{aligned} \mathrm{tr}(x_{\pi,P}w) &= (-1)^i \mathrm{sgn}(\tau_i) \mathrm{tr}\left(w_{\pi,P\backslash p_i,\tau_i} w\right) + (-1)^j \mathrm{sgn}(\tau_j) \mathrm{tr}\left(w_{\pi,P\backslash p_j,\tau_j} w\right) \\ &= S(n,k)k! \left((-1)^i \mathrm{sgn}(\tau_i) + (-1)^j \mathrm{sgn}(\tau_j)\right). \end{aligned}$$

How are τ_i and τ_j related?

$$\begin{aligned} \tau_i^{-1}(l) &= \tau_j^{-1}(l) \text{ for } 1 \le l \le i-1 \\ \tau_i^{-1}(l) &= \tau_j^{-1}(l+1) \text{ for } i \le l \le j-2 \\ \tau_i^{-1}(j-1) &= \tau_j^{-1}(i) \\ \tau_i^{-1}(l) &= \tau_j^{-1}(l) \text{ for } j \le l \le k \end{aligned}$$

In other words, τ_i and τ_j differ by the $(j-i)$-cycle $(i, i+1, \ldots, j-1)$. Thus we have

$$\begin{aligned} \mathrm{tr}(x_{\pi,P}w) &= S(n,k)k! \left((-1)^i \mathrm{sgn}(\tau_i) + (-1)^j \mathrm{sgn}(\tau_j)\right) \\ &= S(n,k)k! \left((-1)^i (-1)^{j-i+1} \mathrm{sgn}(\tau_j) + (-1)^j \mathrm{sgn}(\tau_j)\right) \\ &= S(n,k)k! \cdot \mathrm{sgn}(\tau_j) \left((-1)^{j+1} + (-1)^j\right) \\ &= 0. \end{aligned} \tag{5.2}$$

□

Example 5.4.4. *Let $n = 3, k = 2$ and take $P = \{1, 2, 3\}$. Then*

$$\begin{aligned} x_{12|3,P} &= -112 + 221 + 113 - 331 - 223 + 332, \\ x_{1|23,P} &= -122 + 211 + 133 - 311 - 233 + 322, \text{ and} \\ x_{13|2,P} &= -121 + 213 + 131 - 313 - 232 + 323. \end{aligned}$$

The elements $\{x_{\pi,P}\}$ span $\sqrt{A_{n,1}}$ for all n, and $\{x_{\lambda,P}\}$ and $\{x_{\pi,P}\}$ together span $\sqrt{A_{3,2}}$, but in general their span is some (very) proper subideal of the radical.

The following result gives us a way to "lift" certain elements of the radical.

Proposition 5.4.5. *Fix a subset $P = \{p_1, p_2, \ldots, p_k\} \subseteq [n], k > 1$, and a partition $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$ of n into distinct parts. Define*

$$\bar{x}_{\lambda,P} = \sum_{\sigma \in S_k} sgn(\sigma) \sum_{\pi} \bar{w}_{\pi,P},$$

where $\bar{w}_{\pi,P}(i) = \begin{cases} w_{\pi,P}(i) & 1 \leq i \leq n \\ n+1 & i = n+1 \end{cases}$, and the inside sum is over all ordered partitions $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ with $|\pi_i| = \lambda_{\sigma(i)}$. Then $\bar{x}_{\lambda,P} \in \sqrt{A_{n+1,k+1}}$.

Proof. Given an arbitrary $w = \begin{pmatrix} \nu_1 & \nu_2 & \cdots & \nu_{k+1} \\ s_1 & s_2 & \cdots & s_{k+1} \end{pmatrix}$ we need to show that $\mathrm{tr}(\bar{x}_{\lambda,P} w) = 0$. Suppose $\pi = (\pi_1, \pi_2, \ldots, \pi_k)$ is an ordered partition of $[n]$ with part sizes $\lambda = \lambda_1, \lambda_2, \ldots, \lambda_k$ (not necessarily in that order) such that $\mathrm{tr}(\bar{w}_{\pi,P} w) = 0$. Then $\bar{w}_{\pi,P} w$ must fix $p_1, p_2, \ldots, p_k, p_{k+1}$, where $p_{k+1} = n+1$. How can this happen? First of all, the $k+1$ elements $p_1, p_2, \ldots, p_k, p_{k+1}$ must be in different blocks of ν, say $p_i \in \nu_{\tau(i)}, 1 \leq i \leq k+1$. To get

$$p_1 \longrightarrow \nu_{\tau(i)} \longrightarrow s_{\tau(i)} \longrightarrow \pi_i \longrightarrow p_i,$$

we need $s_{\tau(i)} \in \pi_i, 1 \le i \le k+1$ (here $\pi_{k+1} = \{n+1\}$). So

$$\begin{aligned}
\mathrm{tr}(\bar{x}_{\lambda,P} w) &= \mathrm{tr}\left(\sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \sum_{\substack{\pi \text{ an ordered partition} \\ \text{of } [n] \text{ with } |\pi_i| = \lambda_{\sigma(i)}}} \bar{w}_{\pi,P} w \right) \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \sum_{\substack{\pi \text{ an ordered partition} \\ \text{of } [n] \text{ with } |\pi_i| = \lambda_{\sigma(i)} \\ \text{and } s_{\tau(i)} \in \pi_i}} \mathrm{tr}(\bar{w}_{\pi,P} w) \\
&= \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) S(n+1,k+1)(k+1)! \begin{pmatrix} \#\text{ ordered partitions } \pi \text{ of } [n] \\ \text{with } |\pi_i| = \lambda_{\sigma(i)} \\ \text{and } s_{\tau(i)} \in \pi_i \end{pmatrix} \\
&= S(n+1,k+1)(k+1)! \begin{pmatrix} \#\text{ ordered partitions } \pi \text{ of } [n] \\ \text{with } |\pi_i| = \lambda_i \\ \text{and } s_{\tau(i)} \in \pi_i \end{pmatrix} \sum_{\sigma \in S_k} \mathrm{sgn}(\sigma) \\
&= 0.
\end{aligned}$$

□

Example 5.4.6. *Let* $\lambda = 21, P = \{1,3\}$. *Then*

$$\begin{aligned}
x_{\lambda,P} &= 113 + 131 + 311 - 331 - 313 - 133, \text{ and} \\
\bar{x}_{\lambda,P} &= 1134 + 1314 + 3114 - 3314 - 3134 - 1334.
\end{aligned}$$

Using these this procedure we can construct 48 elements in $\sqrt{A_{4,3}}$. The $x_{\pi,P}$ give us six more. However we know that $\dim\left(\sqrt{A_{4,3}}\right) = 55$, and in any case the elements constructed here are not independent.

5.5 S_n actions

In this last section, we describe two actions of S_n on $\sqrt{A_{n,k}}$. The first is actually a more general action of S_n on each eigenspace of the Gram matrix $M_{n,k}$ (the 0-eigenspace of which is the radical), while the second is specific to $\sqrt{A_{n,k}}$.

Proposition 5.5.1. *S_n acts on each $M_{n,k}$ eigenspace by conjugation.*

Proof. Suppose $M_{n,k}\mathbf{v} = \lambda\mathbf{v}$. Then $(M_{n,k}\mathbf{v})_w = \sum_{w'-w} \mathbf{v}_{w'} = \lambda\mathbf{v}_w$ for all w, where the sum is over all w' such that $w'w$ has k fixed points, i.e., such that w' is connected by an edge to w in the graph $\Gamma_{n,k}$. Now suppose $\sigma \in S_n$. We have

$$\begin{aligned}\left(M_{n,k}\sigma^{-1}\mathbf{v}\sigma\right)_w &= \sum_{w'-w}\left(\sigma^{-1}\mathbf{v}\sigma\right)_{w'}\\ &= \sum_{w'-w}\mathbf{v}_{\sigma^{-1}w'\sigma}\\ &= \sum_{w'-\sigma^{-1}w\sigma}\mathbf{v}_{w'}\\ &= \lambda\mathbf{v}_{\sigma^{-1}w\sigma}.\end{aligned}$$

(We use the fact that $u - v$ in $\Gamma_{n,k}$ if and only if $\sigma^{-1}u\sigma - \sigma^{-1}v\sigma$ in $\Gamma_{n,k}$. This is because uv fixes the k elements $\{s_1, s_2, \ldots, s_k\}$ in $\mathrm{Im}(u)$ if and only if $\sigma^{-1}u\sigma\sigma^{-1}v\sigma = \sigma^{-1}uv\sigma$ fixes the k elements $\{\sigma^{-1}(s_1), \sigma^{-1}(s_2), \ldots, \sigma^{-1}(s_k)\}$ in $\mathrm{Im}(\sigma^{-1}u)$.)

So $M_{n,k}\sigma^{-1}\mathbf{v}\sigma = \lambda\sigma^{-1}\mathbf{v}\sigma$. □

Example 5.5.2. *$M_{3,2}$ has eigenspectrum $\{-3, (-2)^2, -1, 0^5, 1^3, 2^2, 3^3, 4\}$. The corresponding eigenspaces V_λ decompose under the conjugation action of S_3 as follows.*

λ	$\dim V_\lambda$	*decomposition*
-3	1	S^{1^3}
-2	2	S^{21}
-1	1	S^{1^3}
0	5	$2S^{21} \oplus S^{1^3}$
1	3	$S^3 \oplus S^{21}$
2	2	S^{21}
3	3	$S^3 \oplus S^{21}$
4	1	S^3

Proposition 5.5.3. *Let V_0 be the nullspace (zero-eigenspace) of $M_{n,1}$. (So, in particular, $V_0 = \sqrt{A_{n,1}}$.) Let χ be the character of the conjugation action of S_n on V_0. Then, for $\sigma \in S_n$ we have*

$$\chi(\sigma) = (\#\ \textit{fixed points of}\ \sigma) - 1.$$

Proof. Let $\sigma \in S_n$, and suppose σ has j fixed points. Write σ in cycle notation

$$\sigma = (\sigma_1 \ldots \sigma_{r_1})(\sigma_{r_1+1} \ldots \sigma_{r_2}) \cdots (\sigma_{r_{s-1}+1} \ldots \sigma_{r_s}).$$

(Note that $r_s = n$.) Choose as a basis for V_0 the vectors

$$\{\mathbf{w}_i = f_{\sigma_i} - f_{\sigma_{i+1}} | 1 \leq i \leq n-1\},$$

where as before f_i is the constant map taking every element of $[n]$ to i. There are $n-1$ vectors, and they are clearly linearly independent. (Recall Proposition 5.1.1, which states that $V_0 = \sqrt{A_{n,1}}$ is the ideal consisting of all linear combinations of constant maps in which the coefficients sum to zero.) With respect to this basis, and so with respect to any basis, we have

$$\chi(\sigma)=\sum_{i=1}^{n-1}[\mathbf{w}_i]\sigma\mathbf{w}_i\sigma^{-1}=\sum_{i=1}^{n-1}[\mathbf{w}_i]\sigma\mathbf{w}_i,$$

where $[\mathbf{w}_i]$ denotes "the coefficient of $\mathbf{w}_i$ in". What is $[\mathbf{w}_i]\sigma\mathbf{w}_i$?

Writing $\sigma=(\sigma_1\ldots\sigma_{r_1})(\sigma_{r_1+1}\ldots\sigma_{r_2})\cdots(\sigma_{r_{s-1}+1}\ldots\sigma_{r_s})$ we have

$$\begin{aligned}\sigma\mathbf{w}_1 &= \mathbf{w}_2,\sigma\mathbf{w}_2=\mathbf{w}_3,\ldots,\sigma\mathbf{w}_{r_1-2}=\mathbf{w}_{r_1-1}\\ \sigma\mathbf{w}_{r_1-1} &= \sigma\left(f_{\sigma_{r_1-1}}-f_{\sigma_{r_1}}\right)=f_{\sigma_{r_1}}-f_{\sigma_1}=-\left(\mathbf{w}_1+\mathbf{w}_2+\ldots+\mathbf{w}_{r_1-1}\right)\\ \sigma\mathbf{w}_{r_1} &= \sigma\left(f_{\sigma_{r_1}}-f_{\sigma_{r_1+1}}\right)=f_{\sigma_1}-f_{\sigma_{r_1+2}}=\mathbf{w}_1+\mathbf{w}_2+\ldots+\mathbf{w}_{r_1+2}.\end{aligned}$$

The $\sigma\mathbf{w}_{r_1-1}$ term contributes -1 to the sum, while the $\sigma\mathbf{w}_{r_1}$ term contributes +1 to the sum.

This analysis works fine for cycles of length at least two, but what about for fixed points? Suppose for instance that $r_1=1$, i.e., that

$$\sigma=(\sigma_1)(\sigma_2\ldots\sigma_{r_2})\cdots(\sigma_{r_{s-1}+1}\ldots\sigma_{r_s}).$$

Then

$$\sigma\mathbf{w}_1=\sigma\left(f_{\sigma_1}-f_{\sigma_2}\right)=\mathbf{w}_1+\mathbf{w}_2,$$

which contributes +1 to the sum.

In general, fixed points contribute +1, while cycles of length at least two contribute 0. The exception is the very last cycle. If it is a fixed point it contributes nothing, since there is no $\mathbf{w}_n$. Otherwise it contributes -1. So

$$\chi(\sigma)=(\#\text{ fixed points of }\sigma)-1.$$

□

Example 5.5.4. *Let $\sigma=(1)(2)(3)(45)(678)$. Then σ acts on the basis $\{f_1-f_2,f_2-f_3,\ldots,f_7-f_8\}$ of V_0 by the matrix*

$$\begin{pmatrix}1&0&0&0&0&0&0\\0&1&0&0&0&0&0\\0&0&1&0&0&0&0\\0&0&1&-1&1&0&0\\0&0&0&0&1&0&0\\0&0&0&0&1&0&-1\\0&0&0&0&0&1&-1\end{pmatrix},$$

and $tr(\sigma)=1+1+1-1+1+0-1=2=(\#\text{ fixed points of }\sigma)-1$.

Example 5.5.5. *Let* $\sigma = (1254)(37)(6)(8)$*. Then* σ *acts on the basis* $\{f_1 - f_2, f_2 - f_5, \ldots, f_6 - f_8\}$ *of* V_0 *by the matrix*

$$\begin{pmatrix} 0 & 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

and $tr(\sigma) = 0 + 0 - 1 + 1 - 1 + 1 + 1 = 1 = (\#\ \textit{fixed points of}\ \sigma) - 1$*.*

χ decomposes easily into irreducible S_n characters. Since $\chi(\sigma) = (\#\text{ fixed points of }\sigma) - 1$, we have

$$\chi = \chi^{\text{def}} - \mathbf{1},$$

where χ^{def} is the character of the defining representation of S_n. It is well known that $\chi^{\text{def}} = \chi^n + \chi^{(n-1)1} = \mathbf{1} + \chi^{(n-1)1}$, and so we have the following character formula for the S_n conjugation action on $\sqrt{A_{n,1}}$.

Proposition 5.5.6. *The character* χ *of the* S_n *conjugation action on* $\sqrt{A_{n,1}}$ *is given by*

$$\chi = \chi^{(n-1)1},$$

where $\chi^{(n-1)1}$ *is the irreducible* S_n*-character corresponding to the partition consisting of a part of size* $n-1$ *and a part of size 1.*

Finally, we have another action of S_k on the radical.

Proposition 5.5.7. S_n *acts on* $\sqrt{A_{n,k}}$ *by left multiplication.*

Proof. Recall that $\sqrt{A} = \{x \in A | \text{tr}(xy) = 0 \text{ for all } y \in A\}$. Now let $x \in \sqrt{A_{n,k}}, \sigma \in S_n, y \in A_{n,k}$. We want to show that $\text{tr}(\sigma xy) = 0$. We have already shown that S_n acts on $\sqrt{A_{n,k}}$ by conjugation, so $\sigma x \sigma^{-1} \in \sqrt{A_{n,k}}$. But then $\text{tr}(\sigma xy) = \text{tr}(\sigma x \sigma^{-1} \sigma y) = 0$, since $\sigma y \in A_{n,k}$. □

Similarly, S_n acts on $\sqrt{A_{n,k}}$ by right multiplication.

Remark 5.5.8. *For* $\sigma \in S_n$ *we can define operation* $L(\sigma)$ *and* $R(\sigma)$ *on* $A_{n,k}$*, where* $L(\sigma)x = \sigma x$ *(left multiplication) and* $R(\sigma)x = x\sigma$ *(right multiplication). Since the algebra is associative, L and R commute. However, neither commutes with* $M_{n,k}$*. This explains why* S_n *acting on the left or the right does not in general preserve the eigenspaces of* $M_{n,k}$*, for example* $(M_{n,k}x)\sigma \neq M_{n,k}(x\sigma)$*. On the other hand, the matrix* $L(\sigma)R(\sigma^{-1})$ does *commute with* $M_{n,k}$*. This was how we showed that* S_n *acts on the eigenspaces of* $M_{n,k}$ *by conjugation.*

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