2009 Advanced Calculus 142A – Revision problems

These problems should be roughly as hard as the ones on the exam, though I haven't checked very carefully...

1. For each set S below, determine the infimum and supremum, if these exist (and prove that what you say is true!)

$$\{x : x^2 - 3x < 4\} \qquad \{x : |x| + |x-4| \leq 6\} \qquad \{1 + \frac{4}{n} : n \geq 1\}$$

2. Let a, b be distinct real numbers. Prove that there is a rational number in (a, b).

3. Suppose $a_n \leq b_n \leq c_n$ and that the sequences a_n and c_n both converge to some number L. Show that $b_n \to L$.

4. Find the limit of each of the following sequences, if it exists. (Do these directly without using the limit laws).

$$\frac{n-1}{n+1} \qquad \frac{5n^2+n-4}{2n^3+3} \qquad \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}$$

5. Evaluate the limits (and prove your evaluation is correct)

$$\lim_{x\to 1} \frac{x^n - 1}{x-1} \qquad \lim_{x \to 6} \frac{\sqrt{x-2} - 2}{x-6}$$

6. Suppose f is a function on $[a, b]$ which is monotonically increasing. Show that for each $c \in [a, b]$, the limit $\lim_{x\to c^-} f(x)$ exists.

7. Suppose S is a non-empty set, bounded above, with no maximal element. Show that there exists an infinite sequence x_n of real numbers converging to the supremum $s = \sup(S)$.

8. Find the limit

$$\lim_{n\to\infty} \frac{n^n}{n!2^n}$$

9. Show that if f is continuous on $[a, b]$ and is never zero, then its values all have the same sign.

10. Prove that if f is continuous at $x = a$ then whenever x_n is a sequence converging to a, $f(x_n) \to f(a)$. (i.e. the $\epsilon - \delta$ definition of continuity implies Bryant's sequence-based definition).

11. Show that the series $\sum \frac{x^n}{n!}$ converges, for any value of x.

12. Show that if $\sum a_n$ converges absolutely, and b_n is a bounded sequence, then $\sum a_n b_n$ converges absolutely.

13. Determine whether the following series converge or diverge. (Prove your assertion, but you don't need to work out what they converge to if they do!)

$$\sum \frac{n^2-2}{n^3-3} \qquad \sum \frac{n^2-2}{n^4-4}$$

$$\sum (-1)^n \frac{1}{\sqrt{n+2}} \qquad \sum \frac{1}{\sqrt{n+1}} \qquad \sum \frac{(-3)^n}{n.n!}$$