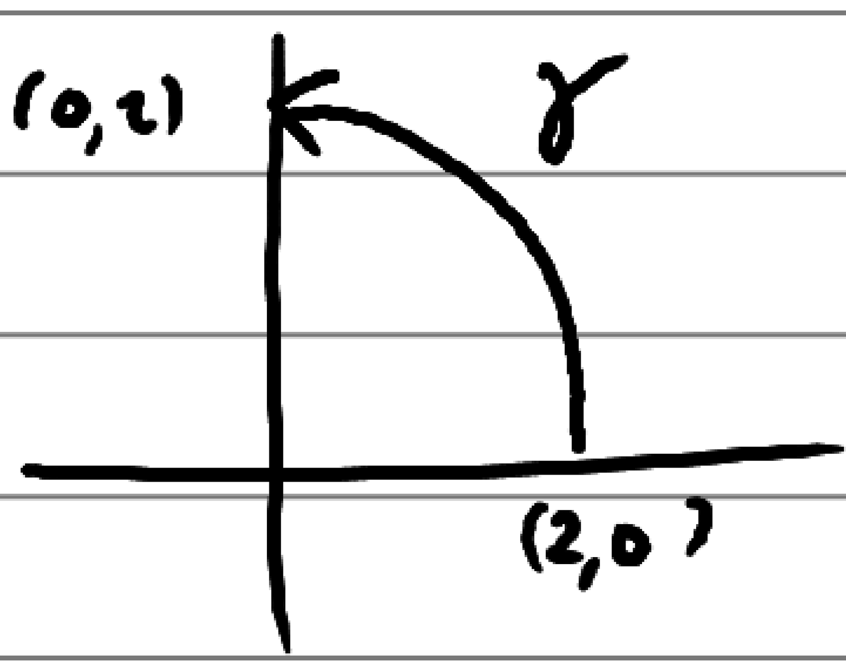
20E Fall 2019 MT 2 Solutions

① 

$$\underline{F}(x,y) = \begin{pmatrix} xy \\ x^2y^2 \end{pmatrix}$$

Parametrize $\quad \underline{s}(\theta) = \begin{pmatrix} 2\cos\theta \\ 2\sin\theta \end{pmatrix}$

$0 \le \theta \le \frac{\pi}{2}$

$$\frac{d\underline{s}}{d\theta} = \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \end{pmatrix}$$

$$\int_\gamma \underline{F}\cdot d\underline{s} = \int_0^{\pi/2} \begin{pmatrix} 4\cos\theta\sin\theta \\ 16\cos^2\theta\sin^2\theta \end{pmatrix} \cdot \begin{pmatrix} -2\sin\theta \\ 2\cos\theta \end{pmatrix} d\theta$$

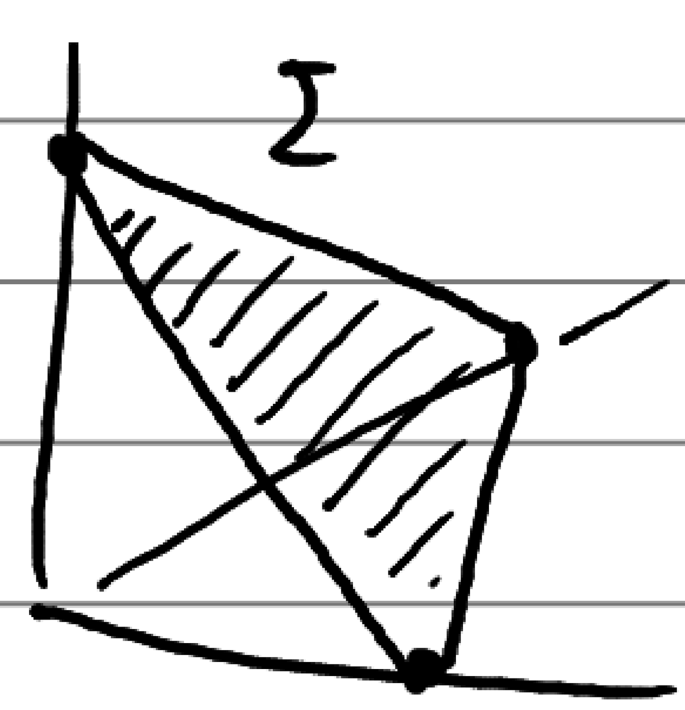
$$= \int_0^{\pi/2} \left(-8\sin^2\theta\cos\theta + 32\cos^3\theta\sin^2\theta\right) d\theta$$

now use $\cos^2\theta = 1-\sin^2\theta$

$$= \int_0^{\pi/2} \left(24\sin^2\theta\cos\theta - 32\sin^4\theta\cos\theta\right) d\theta$$

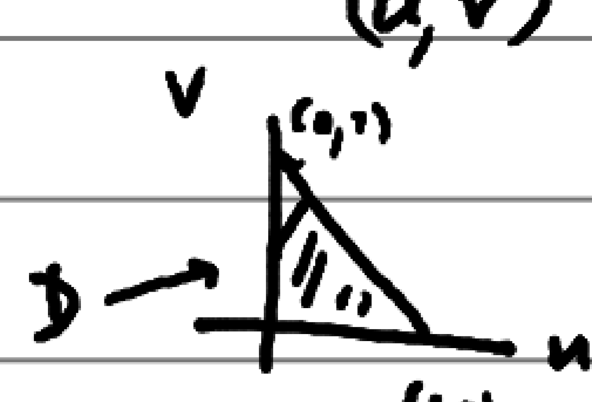
$$= \left[8\sin^3\theta - \frac{32}{5}\sin^5\theta \right]_0^{\pi/2}$$

$$= 8 - \frac{32}{5} = 8/5$$

② 

Parametrize via

$$(u,v) \longmapsto (u, v, 1-u-v)$$

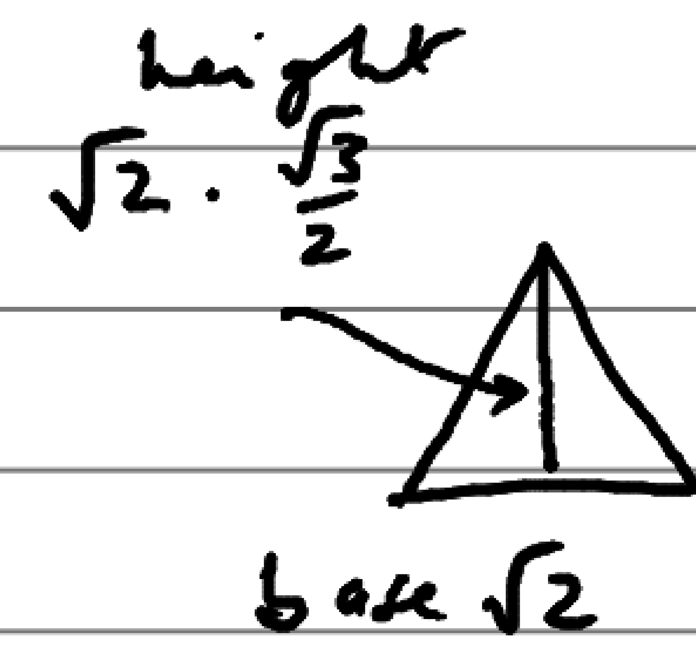


gives $\underline{T}_u \times \underline{T}_v = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$, $\|\cdot\| = \sqrt{3}$

so $\displaystyle\int_\Sigma x^2\, dA = \iint_D u^2 \cdot \sqrt{3}\; du\, dv$

$$= \int_0^1 du \int_0^{1-u} dv \cdot u^2 \cdot \sqrt{3} = \int_0^1 du \cdot \sqrt{3}\, u^2 (1-u)$$

$$= \sqrt{3}\left(\frac{1}{3} - \frac{1}{4}\right) = \frac{\sqrt{3}}{12}.$$



Area of triangle $= \frac{1}{2} \cdot \sqrt{2} \cdot \left(\frac{\sqrt{3}}{2} \cdot \sqrt{2}\right) = \frac{\sqrt{3}}{2}$

$\therefore$ Average is $\frac{\sqrt{3}}{12} \Big/ \frac{\sqrt{3}}{2} = \underline{\underline{\frac{1}{6}}}$

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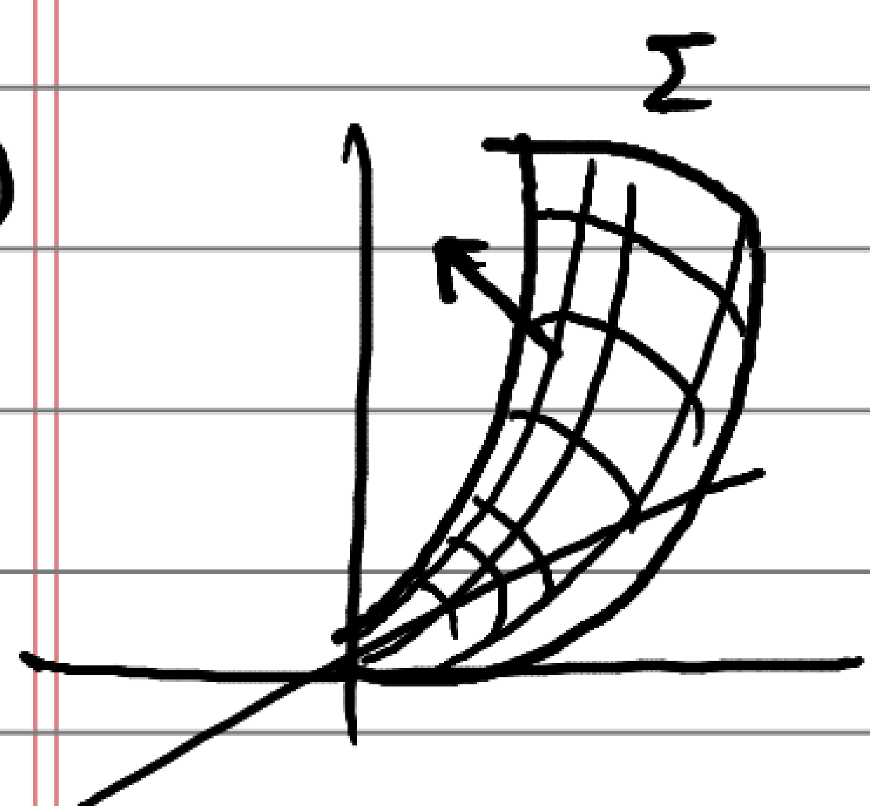


$z = x^2 + y^2$

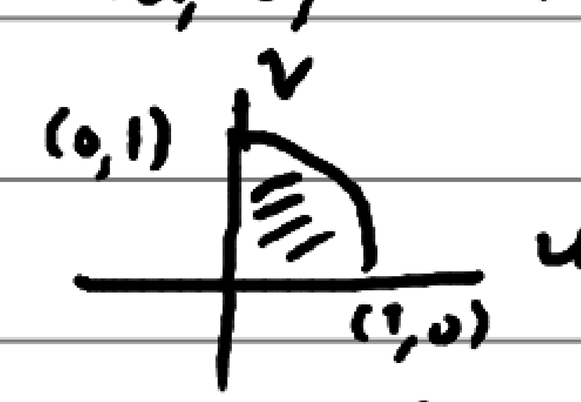
$x, y \geq 0$

$z \leq 1$

Parametrise via

$(u,v) \longmapsto (u, v, u^2+v^2)$

$(u,v) \in D =$



so $\underline{T}_u \times \underline{T}_v = \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix}$ (correct orientation ✓)

Then $\displaystyle\int_\Sigma \begin{pmatrix} x \\ y \\ z \end{pmatrix} \cdot d\underline{A} = \iint_D \begin{pmatrix} u \\ v \\ u^2+v^2 \end{pmatrix} \cdot \begin{pmatrix} -2u \\ -2v \\ 1 \end{pmatrix} du\,dv$

$$= \iint_D -(u^2+v^2)\, du\,dv$$

change to polar coords, $du\,dv = r\,dr\,d\theta$

$$= \int_0^1 dr \int_0^{\pi/2} d\theta \cdot r \cdot (-r^2)$$

$$= \frac{\pi}{2} \cdot \left(-\frac{1}{4}\right) = \underline{\underline{\frac{-\pi}{8}}}$$