1. \[\int_{C} (x^2 - y^2) \, dx + (x^2 + y^2) \, dy \]

\[= \iint_{D} 2x - (-2y) \, dx \, dy \]  

(Green's)

Easier to think of first \( \int_{\gamma} \int_{\gamma} 2x \, dx \, dy \)

\[= \left[ x^2 \right]_{0}^{1} \times \text{length of y-range} = 1 \]

so \( \iint_{D} (2x + 2y) \, dx \, dy = 2 \)

2. Change to spherical coordinates, \( Z = \cos \phi \)

\(dA = \sin \phi \, d\phi \, d\theta\)

\[\iint_{S} z^2 \, dA = \int_{\phi}^{\pi} \int_{\theta}^{2\pi} z^2 \sin \phi \, d\phi \, d\theta\]

\[= 2\pi \times \left[ -\frac{1}{5} \cos^5 \phi \right]_{0}^{\pi} = \frac{2\pi}{5}\]

3. I like to do these using the knowledge of the outward unit normal; at the point \((x, y, z)\),

the outward unit normal is \(\mathbf{n} = \left( \frac{x}{\sqrt{x^2 + y^2 + z^2}}, \frac{y}{\sqrt{x^2 + y^2 + z^2}}, \frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)\)

(it's really the unit normal at \((0, 0, 2)\) on a circle of radius 2)
& with no $z$-coordinate since it points horizontally.

Then we have 

$$\int \mathbf{E} \cdot d\mathbf{A} = \int_{S_1} \left( \mathbf{E} \cdot \hat{\mathbf{n}} \right) dA$$

$$= \int_{S_1} \left( \frac{\pi}{2} \cdot \left( \frac{x}{\sqrt{4}} \right) \right) dA$$

$$= \int_{S_1} \frac{1}{2} (x^2 + y^2) dA$$

Since $x^2 + y^2 = 4$ is constant, 

$$= \int 2 dA = 2 \times \text{area}$$

Slicing & (area of cylinder: imagine unrolling it into a $2\pi r \times h$ rectangle) circumference height 

$$= 16\pi$$

(Alternative: parametric cylinder via 

$$(z, \theta) \mapsto (2\cos \theta, 2\sin \theta, z)$$

$-1 \leq z \leq 1$ 

$0 \leq \theta \leq 2\pi$ \ & compute \ 

$$\iint \mathbf{F} \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) \, d\theta \, dz.$$ 

(we want $\mathbf{r}_\theta \times \mathbf{r}_z$ to give the outward normal: $\mathbf{r}_z$ direction of $z$-increase $\mathbf{r}_\theta$ direction of $\theta$-increase) 

right-hand rule $\Rightarrow \mathbf{r}_\theta \times \mathbf{r}_z$ is \ 

outwards.

In any case $\mathbf{E} = \left( \frac{v}{z} \right)$ clearly shows out of the cylinder, so a positive value of flux must occur.
circle in
$(x, z)$ plane
of radius 3

boundary circle is in fact
oriented clockwise as we see it,
so as to circulate anticlockwise around
the exterior normal vector
(different)
(perspective)

Barycentric method: use Stokes
\[
\int_{\Sigma} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} = \int_{\partial \Sigma} \mathbf{F} \cdot d\mathbf{s}
\]

parametrise the boundary circle by
\[
\theta \mapsto \left( \cos \theta, 3 \cos \theta, 0, -3 \sin \theta \right)
\]

- I put a minus sign in the $z$-coord to make
the clockwise circulation,
- but you could just go anticlockwise
and put the minus sign in at the end, explaining
why!

Then compute
\[
\int_{0}^{2\pi} d\theta \left( \mathbf{F} \cdot \frac{d\mathbf{s}}{dt} \right)
\]

\[
= \int_{0}^{2\pi} \left( \begin{array}{c} 5(-3 \sin \theta) \\ 1 \\ 0 \end{array} \right) \cdot \left( \begin{array}{c} -3 \sin \theta \\ 0 \\ -3 \cos \theta \end{array} \right) d\theta
\]

when $y = 0$, $\mathbf{E}(x, y, z)$ is
\[
\left( \begin{array}{c} 5z \\ 1 \\ 0 \end{array} \right)
\]

in the circle
= \int_0^{2\pi} 45 \sin^2 \theta \, d\theta

= 45\pi \left( \sin^2 \theta = \frac{1}{2} (1 - \cos 2\theta) \right)

\text{Contributes} \pi \quad \text{Contributes} \ 0

to \int_0^{2\pi} \text{ (integral over whole cycle average out to 0.)}

\underline{Fun method:}

Since we're integrating a curl, we can (by using Stoke's theorem twice, essentially) change the surface to another surface with the same oriented boundary, such as the flat disc in the \((x, z)\) plane, of radius 3 whose outward unit normal is just \(\mathbf{j}\).

Thus we want to compute \(\int \mathbf{\nabla} \mathbf{E} \cdot \mathbf{j} \, dA\) over the disc.

The \(\mathbf{j}\)-component of the curl is

\[
\begin{vmatrix}
\frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\
0 & 2xyz & z^2 \\
x^2 & y & 0
\end{vmatrix}
\]

\[
= - (2xy - (2xy + 5)) = +5
\]

Thus we have simply \(\int 5 \, dA = 5 \times \text{area}
\]

\[
\text{disk of radius 3} = 45\pi
\]

(\(\text{Mike forgot the radius 3 in his solution.}\))
0 \quad y = x^3 \quad \int x^2 \, dx - xy \, dy

The "obvious" parametrisation of the curve is "via the $x$-coordinate", i.e. $t \mapsto (t, t^{3/2})$
$0 \leq t \leq 1$ 
This gives you
$$\int_0^1 dt \quad \begin{pmatrix} t^{2/3} \\ -t^{1/2} \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3/2 \end{pmatrix}$$
$$= \frac{d}{dt} \begin{pmatrix} x^2 y \\ -xy \end{pmatrix}$$

But I hate fractional powers!
So an easier calculation will result if we use instead $t \mapsto (t^2, t^3)$, $0 \leq t \leq 1$.
(i.e. $x = t^2, y = t^3$ and satisfying $y^2 = x^3$).
Because then we get
$$\int_0^1 dt \quad \begin{pmatrix} t^7 \\ -t^5 \end{pmatrix} \cdot \begin{pmatrix} 2t \\ 3t^2 \end{pmatrix} = \int_0^1 (2t^8 - 3t^7) \, dt$$
$$= \frac{2}{9} - \frac{3}{8} = \frac{-11}{72}$$

NB: Can't use Green's theorem here - not a closed curve!
You're not expected to find the 'trick' parametrisation above, but it's a good example of a "non-standard" one being useful.
2. Standard Green's theorem is:

\[ \int (x y^2) \, dx + (x^3 + y^4) \, dy = \iint 3x^2 + 2y \, dudv \]

\[ = \iint 3x^2 \, dudv \quad \text{(second term vanishes by symmetry)} \]

\[ = \frac{3}{2} [x^3]_{-1}^1 \times (6 + 7k) \]

\[ = 2 \times 2 = 4 \]

3. I did this in the review lecture:

using \( \mathbf{\hat{n}} = \left( \frac{x}{r} \right) \) we have \( E \cdot \mathbf{\hat{n}} = -x^2 + y^2 + z^2 \)

so need \( \iint (-x^2 + y^2 + z^2) \, dA \).

By symmetry \( \iint x^2 \, dA = \iint y^2 \, dA = \iint z^2 \, dA \)

(at the sphere)

but \( \iint (x^2 + y^2 + z^2) \, dA = \sqrt{r^2} \, dA = 4\pi \)

on unit sphere

\[ = \frac{4\pi}{3} \]

so \( \iint (-x^2 + y^2 + z^2) \, dA = -\frac{4\pi}{3} + \frac{4\pi}{3} + \frac{4\pi}{3} = \frac{4\pi}{3} \)

(by using 'graph' approach, \( z = \sqrt{x^2 + y^2} \), double it!)

(You could also compute \( \iint z^2 \, dA \) on a hemisphere, then...)}
There's a typo in the question:

\[ ds \text{ should be } ds \]

By Stokes, \( \int_C \mathbf{F} \cdot d\mathbf{S} = \int_D \nabla \times \mathbf{F} \cdot \mathbf{k} \, dS \)

\( \nabla \times \mathbf{F} \) is easy:

\[
\begin{pmatrix}
\frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}
\times
\begin{pmatrix}
x^2 \\
y^2 \\
z^2
\end{pmatrix}
= \begin{pmatrix}
x^2 \\
yz \\
0
\end{pmatrix}
\]

So we get \( \int_D \begin{pmatrix}
x^2 \\
yz \\
0
\end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \end{pmatrix} \, dS = 0 \) on the disc.