Solutions
to
Selected Exercises
in Homework #6
\( S7.4 \ #5 \)

(a) \( T_u \times T_v = \begin{vmatrix} i & j & k \\ e^u \cos v & e^u \sin v & 0 \\ -e^u \sin v & e^u \cos v & 1 \end{vmatrix} = (e^u \sin v, -e^u \cos v, e^u) \)

\[ ||T_u \times T_v|| = \sqrt{e^{2u} + e^u} = e^{\sqrt{1 + e^u}} \]

(b) \( \Phi(0, \frac{\pi}{2}) = (0, \frac{\pi}{2}) \), the point on the surface, is \((x_0, y_0, z_0) = \Phi(0, \frac{\pi}{2}) = (0, 1, \frac{\pi}{2})\).

The equation of tangent plane at \((x_0, y_0, z_0)\) is

\[(x-x_0, y-y_0, z-z_0) = 0\]

so the tangent plane at \(\Phi(0, \frac{\pi}{2})\) is

\[(u, v) = (0, \frac{\pi}{2})\]

\[(e^u \sin v, -e^u \cos v, e^u) \cdot (x-x_0, y-y_0, z-z_0) = 0\]

i.e.

\[x + z = \frac{\pi}{2}\]

(c) \( \Phi(D) = \int_0^1 \int_0^\pi ||T_u \times T_v|| \, du \, dv = \int_0^\infty e^{\sqrt{1 + e^u}} \, du \, dv = \int_0^1 \int_0^\infty e^{x^2} \, dx \, dv \)

\[x = e^u\]

use the table in the front page.

Well, the solution in the book is incorrect for part (c).
The surface is indeed a graph of a function: \( z = 1 - x - y \)

So the normal vector to the tangent plane is \((-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1) = (1, 1, 1)\)

Then the area is
\[
A = \iint_D \| (1, 1, 1) \| \, dx \, dy = \sqrt{3} \iint_D dx \, dy,
\]
where
\[D = \{(x, y): x^2 + y^2 \leq 1\}\]

There are at least 2 ways to compute \(\iint_D \, dx \, dy\).

1. This is the same as what you did in Problem #3 last midterm.

Let \( T(u, v) = (v \cos u, \frac{v}{\sqrt{2}} \sin u) \) where \( 0 \leq u \leq 2\pi, 0 \leq v \leq 1 \).

Then the Jacobian is
\[
\frac{\partial (x, y)}{\partial (u, v)} = \begin{vmatrix} v \cos u & \frac{v}{\sqrt{2}} \sin u \\ \frac{v}{\sqrt{2}} \sin u & \frac{v}{\sqrt{2}} \cos u \end{vmatrix} = -\frac{v}{\sqrt{2}}
\]

Now
\[
\iint_D \, dx \, dy = \int_0^{2\pi} \int_0^1 \left| \frac{\partial (x, y)}{\partial (u, v)} \right| \, dv \, du = \int_0^{2\pi} \frac{\sqrt{2}}{\sqrt{2}} \, du = \frac{2\pi}{\sqrt{2}} \left[ \frac{v^2}{2} \right]_0^1 = \frac{\pi}{\sqrt{2}}
\]

\[A = \frac{\sqrt{3} \pi}{\sqrt{2}} = \frac{\sqrt{6} \pi}{2}\]

2. \( D \) is an ellipse. The formula of the area of an ellipse is \(\pi ab\).

Hence, \( A = \frac{\sqrt{3}}{\sqrt{6}} (\pi \cdot \frac{1}{\sqrt{2}}) = \frac{\sqrt{6} \pi}{2} \)
The surface under consideration is a part of the unit sphere that sits inside the cylinder $x^2 + y^2 = z$.

$x^2 + y^2 = z$ is indeed a cylinder centered at $\left(\frac{1}{2}, 0, 0\right)$ with radius $\frac{1}{2}$ because $x^2 + y^2 = \frac{1}{4}$ implies $(x-\frac{1}{2})^2 + y^2 = \frac{1}{4}$.

Note that $A(S_1) = \text{Area } \odot + \text{Area } \ominus$ and $A(S_2) = \text{Unit sphere } - S_1$.

As $\text{Area } \odot = \text{Area } \ominus$ by symmetry, we have $A(S_1) = 2 \left(\text{Area } \odot\right)$.

The surface $\odot$ sits on the upper hemisphere which is a graph of the function $z = \sqrt{1-x^2-y^2}$ (i.e., $z = 1-x^2-y^2$, $\geq 0$).

Apply implicit differentiation to (x)

A normal vector to the surface $\odot$ at $(x, y, z)$ is $\left(-\frac{\partial z}{\partial x}, -\frac{\partial z}{\partial y}, 1\right) = \left(-\frac{x}{\sqrt{1-x^2-y^2}}, -\frac{y}{\sqrt{1-x^2-y^2}}, 1\right)$.

Now

$\text{Area } \odot = \iint_D \sqrt{\left(\frac{x}{\sqrt{1-x^2-y^2}}, \frac{y}{\sqrt{1-x^2-y^2}}\right)^2 + 1} \, dx \, dy$

$= \iint_D \frac{1}{2} \sqrt{x^2 + y^2} \, dx \, dy$

$= \iint_D \frac{1}{\sqrt{1-x^2-y^2}} \, dx \, dy$. 

"D" is the region in the $xy$-plane defined by $x^2 + y^2 \leq 1$. 

"D" is the region in the $xy$-plane defined by $x^2 + y^2 \leq 1$.
Write \( D = \{ (x, y) : x^2 + y^2 \leq x^2 \} \) in polar coordinate.

\[ D = \{ (r, \theta) : 0 \leq r \leq r \cos \theta, -\pi \leq \theta \leq \pi \} = \{ (r, \theta) : 0 \leq r \leq \cos \theta, -\pi \leq \theta \leq \pi \}. \]

Then 
\[
\text{Area}(\Omega) = \int_{-\pi/2}^{\pi/2} \int_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} r \, dr \, d\theta = \int_{-\pi/2}^{\pi/2} \left[ \frac{1}{2} \left( -\sqrt{1-r^2} \right)^2 \right]_{-\sqrt{1-r^2}}^{\sqrt{1-r^2}} \cos \theta \, d\theta
\]
\[
= \int_{-\pi/2}^{\pi/2} \left( 1 - \sqrt{1 - \cos^2 \theta} \right) \, d\theta = \int_{-\pi/2}^{\pi/2} 1 \, d\theta - \int_{-\pi/2}^{\pi/2} |\sin \theta| \, d\theta = \pi - 2 \int_{0}^{\pi/2} \sin \theta \, d\theta = \pi - 2.
\]

Then 
\[ A(S_1) = 2(\pi - 2), \quad \text{so} \quad A(S_2) = 4\pi - 2(\pi - 2) = 2(\pi + 2). \]

Hence 
\[ \frac{A(S_2)}{A(S_1)} = \frac{\pi + 2}{\pi - 2}. \]
From elementary geometry, the surface area of this cone is $\pi rl$.

(a) If we know the radius is $r$, then $h = mr$, so

$$l = \sqrt{r^2 + h^2} = r\sqrt{1+m^2}.$$  

From above, the area is $\pi r^2 \sqrt{1+m^2}$. Thus the truncated cone has the surface area $\pi (r_2^2 - r_1^2) \sqrt{1+m^2}$.

(b) The truncated cone can be parametrized as

$$\Phi(u,v) = (mv\cos u, mv\sin u, v)$$

where $0 \leq u \leq 2\pi$, $r_1 \leq v \leq r_2$. Then $T_u \times T_v = (v\cos u, v\sin u, mv)$.

Then

$$\text{Area} = \int_{r_1}^{r_2} \int_0^{2\pi} \sqrt{1+m^2} dv du = \sqrt{1+m^2} 2\pi \int_{r_1}^{r_2} v dv = 2\pi \sqrt{1+m^2} \left[ \frac{v^2}{2} \right]_{r_1}^{r_2} = \pi \left( r_2^2 - r_1^2 \right) \sqrt{1+m^2}$$
The plane can be expressed as a graph of the function \( z = 6 - 2x - 3y \), so the normal vector we want is \((2, 3, 1)\). The projection of the plane on the \(xy\)-plane is given by \(2x + 3y = 6\). Because \(S\) is the above plane, the integrand is \(3x - 2y + 2 = 3x - 2y + 6 - 2x - 3y = 6 + x - 5y\).

Then \(\iint_S (3x - 2y + 2)\,dS = \int_0^2 \int_0^{\frac{3-3y}{2}} (6 + x - 5y) \|(2, 3, 1)\|\,dx\,dy = \int_0^2 \int_0^{\frac{3-3y}{2}} (6 + x - 5y) \sqrt{14}\,dx\,dy = 11\sqrt{14}\).
S 7.5 #5 \( S : \Phi(u,v) = (u+v, u-v, u^2) \)

(a) \( S \) is in the graph of \( x^2 - y^2 = 4z \) because \( (u+v)^2 - (u-v)^2 = 4uv \). (\( T_x \times T_y \))

(b) As \( S \) can be described as a graph of a function, the desired normal vector is \( \left( \frac{-\partial z}{\partial x}, \frac{-\partial z}{\partial y}, 1 \right) = \left( -\frac{x}{2}, \frac{y}{2}, 1 \right) \). So \( \left\| \left( -\frac{x}{2}, \frac{y}{2}, 1 \right) \right\| = \frac{1}{2} \sqrt{x^2 + y^2 + 4} \).

We're interested in the graph \( S \) over the region \( x^2 + y^2 \leq 1 \), i.e.

\[-\sqrt{1-y^2} \leq x \leq \sqrt{1-y^2}, \quad -1 \leq y \leq 1.\]

Then

\[
\iint_S x \, dS = \iint_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \left\| T_x \times T_y \right\| \, dx \, dy = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} x \sqrt{x^2 + y^2} \, dx \, dy = 0 \text{ because the integrand is an odd function of } x \text{ and the integration limits in } x \text{ are symmetric.} \]
1: Because the integrand is $xy$ and in the region $R$, the coordinate $y$ is 0, $\iint xy \, dS = 0$

2: The region can be described by $y \leq x \leq 1$, $0 \leq y \leq 1$, which lies on the plane $z = 0$. Then $T_x \times T_y = (1, 0, 1)$.

Then $\iint xy \, dS = \iint xy (1, 0, 1) \, dx \, dy = \frac{1}{8}$

3: The surface $S_3$ can be described as $z = 1 - x$ where $0 \leq x \leq 1$, $0 \leq y \leq 1$.

The normal vector $T_x \times T_y = (1, 0, 1)$.

So $\iint xy \, dS = \iint xy \sqrt{2} \, dx \, dy = \frac{\sqrt{2}}{8}$

4: The surface $S_4$ is $\bar{S}(u,v) = (u, u, v)$ where $0 \leq v \leq 1$, $0 \leq u \leq 1$.

We have $T_u \times T_v = (1, 1, 0)$. Then $\iint xy \, dS = \iint u \, du \, dv = \frac{\sqrt{2}}{12}$

$\therefore \iint xy \, dS = \frac{1}{8} + \frac{\sqrt{2}}{8} + \frac{\sqrt{2}}{12} = \frac{3 + 5\sqrt{2}}{24}$
1: The surface is the graph \( z = 1 - x^2 - y^2 \) with \( z \geq 0 \).

Then \( \mathbf{T}_x \times \mathbf{T}_y = (2x, 2y, 1) \) (it points upward thus is outer normal)

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint (2x, 2y, 2) \cdot (2x, 2y, 1) \ dxdy
\]

\[
= \iint 4x^2 + 4y^2 + 2 \ dxdy = \iint 4x^2 + 4y^2 + 2 - x^2 - y^2 \ dxdy = \iint 4x^2 + 4y^2 \ dxdy
\]

Because the projection of 1 on \( \mathbf{xy} \)-plane is \( \mathbf{r}^2 \leq 1 \), use the polar coordinate to get

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \int_0^{2\pi} \int_0^1 (4r^2) \ r \ dr \ d\theta = \frac{5\pi}{2}
\]

2: The surface is \( z = 0 \). Then \( \mathbf{T}_x \times \mathbf{T}_y = (0, 0, 1) \), so

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint \mathbf{(x, y, 0)} \cdot (0, 0, 1) = 0.
\]

\[
\therefore \iint_S \mathbf{F} \cdot d\mathbf{S} = \frac{5\pi}{4}
\]
S7.6 #2

The surface is \( \Phi(u,v) = (2 \sin u, 3 \sin u, v) \), \( 0 \leq u \leq 2\pi \), \( 0 \leq v \leq 1 \).

Then \( T_u \times T_v = (-3 \sin u, -2 \cos v, 0) \). So we have

\[
\iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \left( 2 \sin u, 3 \cos u, v \right) \cdot (-3 \sin u, -2 \cos v, 0) \, du \, dv
\]

\[
= \int_0^{2\pi} \int_0^1 -3 \sin u \, du \, dv = -12\pi
\]

S7.6 #5

Heat flux is \( k \mathbf{V} \cdot \mathbf{T} = 1 \cdot (6x, 0, 6z) \).

\( \Phi(u,v) = (\sqrt{2} \cos u, v, \sqrt{2} \sin u) \); \( 0 \leq u \leq 2\pi \), \( 0 \leq v \leq 2 \).

Then \( T_u \times T_v = (-\sqrt{2} \cos u, 0, -\sqrt{2} \sin u) \).

\[
\iint_S (k \mathbf{V} \cdot \mathbf{T}) \cdot d\mathbf{S} = \iint_D \left( -6 \sqrt{2} \cos u, 0, 6 \sqrt{2} \sin u \right) \cdot \left( -\sqrt{2} \cos u, 0, -\sqrt{2} \sin u \right) \, du \, dv
\]

\[
= \int_0^{2\pi} \int_0^2 6 \sqrt{2} \, du \, dv = 48 \sqrt{2} \pi
\]
The upper hemisphere is a graph of the function \( z = \sqrt{x^2 + y^2} \),
i.e. \( z = \sqrt{x^2 + y^2}, \quad z > 0 \).

By implicit differentiation, \( T_x \times T_y = \left( \frac{x}{z}, \frac{y}{z}, 1 \right) \). The projection of \( S \)
on to \( xy \)-plane is \( x^2 + y^2 \leq 9 \).

Then \( \iint_S F \cdot dS = \iint_D \left( x, y, z(\sqrt{x^2 + y^2}) \right) \cdot \left( \frac{x}{z}, \frac{y}{z}, 1 \right) \, dx \, dy \)

\[ = \iint_D \frac{1}{2(x^2 + y^2)} \left( x^2 + y^2 + 2(x^2 + y^2)^2 \right) \, dx \, dy = \iint_D \frac{9}{2(x^2 + y^2)} \, dx \, dy = \iint_D \frac{9}{\sqrt{9 - r^2}} \, dr \, d\theta \]

"(x, y, z) is on the sphere of radius 3."

Polar coordinate \( r \)

\[ = \int_0^{2\pi} \int_0^3 \frac{9r}{\sqrt{9 - r^2}} \, dr \, d\theta = 54\pi \]

(b) By symmetry, we get \( \iint_{\text{sphere, radius 3}} F \cdot dS = 2(54\pi) = 108\pi \)
\[ \nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \frac{y}{x} & -\frac{x}{y} & 2x^2y^2 \end{vmatrix} = \left( 2x^2y^2, -3x^2y^2, 1 \right). \]

Surface \( x^2 + y^2 + z^2 = 1, \ z \geq 0 \) is indeed a graph of a function, then by implicit differentiation:

\[ T_x \times T_y = \left( \frac{x}{z^2}, \frac{y}{z^2}, 1 \right). \]

Projection of \( S \)

Then

\[
\int_S (\nabla \times F) \cdot ds = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (2x^2y^2 - 3x^2y^2 + 2) \, dy \, dx
\]

Note that \( g(y) = 2x^2y \) and \( h(y) = 3x^2y^3 \) are odd functions of \( y \) and the integration limits are symmetric in \( y \). Then

\[
\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} g(y) \, dy = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} h(y) \, dy = 0.
\]

Hence

\[
\int_S (\nabla \times F) \cdot ds = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 2D \, dy \, dx = 2\pi.
\]
S: \( x^2 + y^2 + z^2 \leq 1, \quad z > 0 \) is indeed a graph of a function. 

Then \( \mathbf{T}_x \times \mathbf{T}_y = \left( \frac{x}{z}, \frac{y}{z}, 1 \right) \).

Now \( \iint_S F \cdot dS = \iint_D (x + 3y^5, y + 10x^2, z - xy) \cdot (\frac{x}{z}, \frac{y}{z}, 1) \, dx \, dy \).

\[
= \iint_D \left( \frac{x^2}{z} + \frac{3y^5}{z} + \frac{y}{z} + 10x^2 + \frac{z}{z} - xy \right) \, dx \, dy = \iint_D \left( \frac{x^2+y^2+z^2}{z} + \frac{3y^5}{z} + 10x^2 - xy \right) \, dx \, dy.
\]

\[
= \iint_D \frac{1}{2} \, dx \, dy + \iint_D \left( \frac{3y^5}{z} - xy \right) \, dy \, dx + \iint_D 10xz \, dx \, dy.
\]

Note that \( z = \sqrt{1-x^2-y^2} \) is an even function in both \( x \) and \( y \), and a product of an odd function and an even one is odd. We have \( g(y) = \frac{3y^5}{z} - xy \) is odd in \( y \) and \( h(x) = 10xz \) is odd in \( x \). By symmetry of the integration limits,

\[
\iint_D g(y) \, dy \, dx = \iint_D h(x) \, dx \, dy = 0.
\]

Thus \( \iint_S F \cdot dS = \iint_D \frac{1}{2} \, dx \, dy \).

Polar coord., \( 2\pi \times \frac{1}{2} \int_0^1 \int_0^{\sqrt{1-r^2}} \, rd\theta \, dr = 2\pi \).