Vector Calculus 20E, Spring 2013, Lecture A, Final exam

Three hours, eight problems. No calculators allowed.
Please start each problem on a new page.
You will get full credit only if you show all your work clearly.
Simplify answers if you can, but don’t worry if you can’t!

1. Let \( \gamma \) be the closed curve given by the equations \( x = t^2 - t, \ y = 2t^3 - 3t^2 + t \) for \( 0 \leq t \leq 1 \). Using Green’s theorem, find the area enclosed by the curve \( \gamma \).

2. Find the integral \( \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} \) where \( \mathbf{F} = yi + xj + zk \) and \( \gamma \) is the helical curve \( x = 2\cos t, \ y = 2\sin t, \ z = t \) for \( 0 \leq t \leq 2\pi \), oriented in the direction of increasing \( t \).

3. Find the integral \( \int_{\Sigma} y^2 \ dA \), where \( \Sigma \) is the part of the cylinder \( x^2 + y^2 = 4 \) lying between the planes \( z = 0 \) and \( z = x + 3 \).

4. Let \( D \) be the standard unit disc in the \( xy \)-plane, and let \( \Sigma \) be the part of the graph of the function \( z = xy \) lying over the domain \( D \). Find the surface area of \( \Sigma \).

5. Let \( \Sigma \) be the hemisphere \( x^2 + y^2 + z^2 = 16, \ z \geq 0 \), oriented with the upward normal, and let \( \mathbf{F} \) be the vector field \( (x^2 + z)i + 3xyzj + (2xz)k \). Compute the integral \( \int_{\Sigma} (\nabla \times \mathbf{F}) \cdot d\mathbf{A} \).

6. Find the flux of the vector field \( \mathbf{F} = x^2yi + z^8j - 2xyzk \) out of the surface of the standard unit cube \( 0 \leq x, y, z \leq 1 \) in \( \mathbb{R}^3 \).

7. Find the integral \( \int_{\gamma} \mathbf{F} \cdot d\mathbf{s} \) where \( \mathbf{F} = xi + yj + zk \) and \( \gamma \) is the oriented curve given by \( (\sin^2 t \cos t, \cos^2 t \sin t, (t - \pi)^4) \) \( 0 \leq t \leq 2\pi \).

8. One of the two vector fields

\[
\mathbf{F} = 3x^2yi + x^3j + 5k \\
\mathbf{G} = (x + z)i + (z - y)j + (x - y)k
\]

is conservative, and the other is not. Which is which? Find a potential for the conservative one.