1. Homotopy and the properties of the fundamental group

Homotopy

1. Show that any non-surjective map $f : X \to S^n$ is homotopic to a constant map.

2. Let $f, g : X \to S^n$ be such that for any $x \in X$, $f(x)$ and $g(x)$ are not antipodal points on the sphere. Show that $f \simeq g$.

3. Show that when n is *odd*, the *antipodal map* $S^n \to S^n$, given by negation of unit vectors $x \mapsto -x$, is homotopic to the identity map of S^n.

4. A space which is homotopy-equivalent to a point is called *contractible*. Show that a space is contractible if and only if its identity map is homotopic to a constant map.

5. The *Möbius strip* M is defined as $I \times I$ quotiented by the relation $(x, 0) \sim (1 - x, 1), \forall x \in I$. Prove that $S^1 \times I$ is homotopy-equivalent to the Möbius strip M.

6. Show that $\mathbb{R}^3 - S^1$ (the complement of the unit circle in the (x, y)-plane) is homotopy-equivalent to the one-point union (obtained by identifying one point from each) $S^1 \vee S^2$.

7. Classify the capital letters of the alphabet up to homeomorphism and up to homotopy-equivalence! (Assume that $S^1, S^1 \vee S^1$ and a point are not homotopy-equivalent to one another.)

8. (Tricky but important!) Let $f, g : S^1 \to X$ be two maps from the circle to a topological space X. Define a space $P = X \cup_f B^2$ by "attaching a disc along f": form the disjoint union $X \amalg B^2$ and then identify each point $x \in S^1 = \partial B^2$ with its image $f(x) \in X$. Define $Q = X \cup_g B^2$ similarly. Prove that if $f \simeq g$, then $P \simeq Q$; thus, "the homotopy type of $X \cup_f B^2$ depends only on the homotopy class of the attaching map".

Properties of the fundamental group

9. Let X be a path-connected, simply-connected (having trivial fundamental group) space, and let x, y be points of X. Show that all paths from x to y are homotopic rel $\{0, 1\}$.

10. Let X and Y be topological spaces, let A a subspace of X and let $f : A \to Y$ be a map. A map $F : X \to Y$ is said to be an *extension* of f if its restriction to A is given by f. Show that the fundamental group of a path-connected space X is trivial if and only if every continuous map $f : S^1 \to X$ has an extension to a continuous map $F : B^2 \to X$.

11. Show that $\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0)$.

12. Let N and S be the poles of the sphere S^n. Supposing that $n \geq 2$, prove that any path in S^n may be written as a composite of finitely many paths, each of which is contained in $S^n - \{N\}$ or $S^n - \{S\}$, and consequently that $\pi_1(S^n) = 1$ for $n \geq 2$.