10. Universal coefficient theorem

1. Identify the following abelian groups up to isomorphism.

(a). $\mathbb{Z}_m \otimes \mathbb{Z}_n$

(b). $\mathbb{Z}_{60}^4 \otimes (\mathbb{Z}_{24}^3 \oplus Z_8^4 \oplus \mathbb{Z}_{120})$

(c). $\mathbb{Z}_n \otimes \mathbb{Q}$

(d). $(\mathbb{Z} \oplus \mathbb{Z}_n) \otimes (\mathbb{Q}/\mathbb{Z})$

2. (a). Compute $\mathrm{Tor}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_4)$.

(b). Compute $\mathrm{Ext}(\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3, \mathbb{Z} \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_5)$.

3. Recall the calculation of the integer homology of $\mathbb{R}P^n$ by means of a cellular chain complex. Compute the following homology groups directly from the chain complex (and check them by means of the universal coefficient theorem).

(a). $H_*(\mathbb{R}P^n; \mathbb{Z}_2)$

(b). $H_*(\mathbb{R}P^n; \mathbb{Z}_3)$

(c). $H^*(\mathbb{R}P^n; \mathbb{Z}_6)$

4. Show that the first cohomology group of any space is free abelian.

5. Show that for any space, the rational homology and cohomology can be computed via

$$H_*(X;\mathbb{Q}) = H_*(X;\mathbb{Z}) \otimes \mathbb{Q} \qquad H^*(X;\mathbb{Z}) = \mathrm{Hom}(H_*(X;\mathbb{Z}), \mathbb{Q}).$$

6. Explain how to construct a space X having homology groups $H_i(X;\mathbb{Z})$ given by $\mathbb{Z}, \mathbb{Z}_6, \mathbb{Z}_{12}, \mathbb{Z} \oplus \mathbb{Z}_4$ in dimensions $i = 0 \ldots 3$, and zero otherwise. Compute its cohomology groups $H^*(X;\mathbb{Z})$.

7. Compute the homology $H_*(\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z}_2)$.

8. Compute the integral homology $H_*(\Sigma\mathbb{R}P^2 \times \mathbb{R}P^2; \mathbb{Z})$. ($\Sigma X$ denotes the suspension of X, namely $X \times I$ with the ends $X \times \{0\}$ and $X \times \{1\}$ each collapsed to a (different) point).

9. Compute the integral homology $H_*(\mathbb{R}P^2 \times \mathbb{R}P^3; \mathbb{Z})$.

10. Show that if G is a topological group, then $H_*(G)$ has a natural (possibly non-commutative) algebra structure. Show also that there is a natural action of G on $H_*(G)$, but that this action factors through the homomorphism $G \to \pi_0(G)$, so that it's trivial if G is path-connected.