11. Homological algebra

1. Let A be an abelian group. Show that the kernel of the natural map $A \to A \otimes \mathbb{Q}$ (sending a to $a \otimes 1$) is the subgroup of torsion elements (elements of finite order) $\mathrm{Tors}(A)$. (Don't confuse Tors with Tor!)

2. Suppose $A \to B$ is an injective map of abelian groups. Show that the natural map $A \otimes \mathbb{Q} \to B \otimes \mathbb{Q}$ is also injective. This shows that tensoring with $\mathbb{Q}$ is left exact as well as right exact; that is, $\mathbb{Q}$ is *flat* as a $\mathbb{Z}$-module.

3. Find a free resolution for $\mathbb{Q}$ as a $\mathbb{Z}$-module. (Hint. $\frac{1}{n!}$.)

4. Let A be an abelian group. Compute $\mathrm{Tor}(\mathbb{Q}, A)$. Now consider the group $\mathbb{Q}/\mathbb{Z}$, given by the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0.$$

(It can be thought of as the group of roots of unity, by identifying $[p/q]$ with $e^{2\pi i p/q}$). Compute $\mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A)$.

5. Extend the table of values of $\mathrm{Tor}(A, B)$ and $\mathrm{Ext}(A, B)$ to the cases where A and B may be $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$.

6. Let $R = \mathbb{Z}[x, y]$ be the polynomial ring in two variables. Consider the quotients $M = R/(x - y)$ and $N = R/(x, y)$ as R-modules. Construct free resolutions of M and N and thereby compute $\mathrm{Ext}^*_R(M, M)$, $\mathrm{Ext}^*_R(M, N)$, $\mathrm{Ext}^*_R(N, M)$, and $\mathrm{Ext}^*_R(M, M)$.

7. Consider the exterior algebra Λ^*, generated by anti-commuting symbols $dx_1, dx_2, \ldots, dx_n$, over some field k. Show that taking "wedge product with dx_1" as differential gives a chain complex

$$0 \to \Lambda^0 \to \Lambda^1 \to \cdots \to \Lambda^n \to 0$$

with trivial homology. What happens if dx_1 is replaced by an arbitary non-zero element $v \in \Lambda^1$?

8. Consider the group ring $R = \mathbb{Z}[\mathbb{Z}_2]$, and let x denote the non-trivial element of $\mathbb{Z}_2$. There is an action of R on $\mathbb{Z}$, given by letting x act by multiplication -1; call this module M. Construct a free resolution of M and use it to compute the groups $\mathrm{Tor}^R_*(M, M)$.

9. Show that the functor $\mathrm{Tor}^R_*(-, -)$ is symmetric, for any commutative ring R, as follows. Given modules M and N, take free resolutions M_* and N_*. There is a chain map $M_* \to M$, if we view M as a complex whose only non-zero chain group is M in dimension zero. Tensoring with the complex N_* gives us a chain map $\psi : M_* \otimes_R N_* \to M \otimes N_*$. By considering the exact sequence of complexes

$$0 \to (Z_*, 0) \to (N_*, d) \to (B_{*-1}, 0) \to 0$$

and a technique similar to that of the Künneth theorem, conclude that ψ gives an isomorphism on homology. By now switching the roles of M and N, conclude that Tor is symmetric.

10. An exact sequence of abelian groups

$$0 \to A \to B \to C \to 0$$

is called an *extension* of C by A. (This is the topologist's convention: we think of B as resembling the total space of a bundle over the base space C with fibre A, in other words, as "a family of 'A's parametrised by C". Algebraists point out that it is A which is a subgroup of B, so they call B

an extension of A by C! Admittedly, logic is on their side, but the first convention is more useful and is prevalent these days.) Extensions of C by A form a category in a fairly obvious way; in particular an *equivalence* is an isomorphism $\theta : B \to B'$ making a commuting diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow{i} & B & \xrightarrow{p} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \mathrm{id}_C} & & \\ 0 & \longrightarrow & A & \xrightarrow{i'} & B' & \xrightarrow{p'} & C & \longrightarrow & 0. \end{array}$$

Follow the steps below to prove that *the group* $\operatorname{Ext}(C, A)$ *classifies such extensions up to equivalence*, and hence deserves its name!

(a). Show how to associate to an (equivalence class of) extension $E = (0 \to A \to B \to C \to 0)$ a (well-defined) characteristic element $c(E) \in \operatorname{Ext}(C, A)$. (Hint: let $0 \to F_1 \to F_0 \to C \to 0$ be a free resolution of C; then construct a map $F_1 \to A$.)

(b). Show how to associate to an element $c \in \operatorname{Ext}(C, A)$ an equivalence class of (group B and) extension $E(c) = (0 \to A \to B \to C \to 0)$.

(c). Check that these two procedures are mutually inverse.

(d). Show that a split extension corresponds to the zero element of $\operatorname{Ext}(C, A)$. We can therefore view the class $c(E) \in \operatorname{Ext}(C, A)$ as measuring the *obstruction to splitting* of E.

(e). (If you're still having fun!) We know that $\operatorname{Ext}(C, A)$ is itself an abelian group. Work out how to define the *sum* $E+F$ of two extensions of C by A in such a way that $c(E+F) = c(E)+c(F)$, so that $\operatorname{Ext}(C, A)$ may be identified with the *abelian group* of equivalence classes of extensions.

It's a remarkable fact that higher Ext-groups classify equivalence classes of longer exact sequences: for example, $\operatorname{Ext}^2(C, A)$ classifies equivalence classes of sequence $0 \to A \to X_1 \to X_2 \to C$. Moreover, there is a multiplication (or composition) $\operatorname{Ext}^m(C, B) \times \operatorname{Ext}^n(B, A) \to \operatorname{Ext}^{m+n}(C, A)$ called the *Yoneda product* obtained by splicing them!