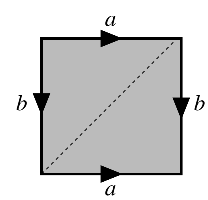
12. Cohomology ring

Cohomology ring

1. Glue two triangles together to make the torus $T = S^1 \times S^1$ as shown below. Write down singular chains p, a, b, t representing generators of its homology groups in dimensions 0,1,1,2. Now let $\alpha, \beta \in H^1$ be the Poincaré duals of $[a], [b] \in H_1$. Show directly from the definition in *singular* (co)homology that $\alpha \cup \beta$ is a generator of H^2 and that the cohomology ring of T is an exterior algebra on two generators of degree 1.



2. Using a similar argument in singular (co)homology, calculate the cohomology ring of the closed orientable surface Σ_g of genus g.

3. Describe the cohomology rings $H^*(X \amalg Y), H^*(X \vee Y)$ in terms of $H^*(X)$ and $H^*(Y)$.

4. Show that $S^1 \times S^1$ and $S^1 \vee S^1 \vee S^2$ are not homotopy-equivalent.

5. Let ΣX be the suspension of a space X, and let U, V be the open sets obtained as the complements of the two suspension points. Show that $H^*(\Sigma X, U) \cong \tilde{H}^*(\Sigma X)$. By considering the cup product $H^*(\Sigma X, U) \times H^*(\Sigma X, V) \to H^*(X, U \cup V)$, deduce that the cup product of any two cohomology classes of positive degree on ΣX is zero.

6. Suppose X is a space with a cover by contractible open sets $U_1, \ldots U_n$. Show that the cup product of any n elements of $H^{>0}(X)$ is zero, and hence that projective n-spaces cannot be covered by fewer than $n+1$ contractible open sets.

7. Show by generalising the method of question 1 that the cohomology ring of the n-torus T^n is an exterior algebra generated by H^1. In other words, there is a basis $\alpha_1, \ldots, \alpha_n$ for $H^1(T^n; \mathbb{Z})$ such that

$$H^*(T^n; \mathbb{Z}) = \Lambda^*[\alpha_1, \ldots, \alpha_n].$$

(Recall: this is the free associative (but non-commutative) algebra in the α_is, with relations $\alpha_i \alpha_j = -\alpha_j \alpha_i$. We write $\wedge$ for the product in this algebra. The dimension of H^k is then $\binom{n}{k}$ with basis $\{\alpha_{i_1} \wedge \alpha_{i_2} \wedge \cdots \wedge \alpha_{i_k} : i_1 < i_2 < \cdots < i_k\}$.)

8. Assume the fact that $H^*(\mathbb{C}P^n; \mathbb{Z}) = \mathbb{Z}[\alpha]/(\alpha^{n+1})$, where α has degree 2, and $H^*(\mathbb{R}P^n; \mathbb{Z}_2) = \mathbb{Z}_2[\beta]/(\beta^{n+1})$, where β has degree 1. (These are "truncated polynomial rings".) Show that if $k < n$, there is no retraction from $\mathbb{C}P^n$ to its standard submanifold $\mathbb{C}P^k$, and similarly for $\mathbb{R}P^n$.

Compact support and direct limits

9. Show that (working with cellular chain complexes) $H_c^*(X \times \mathbb{R}) \cong H_c^{*-1}(X)$ for any space X. This is the analogue for compactly-supported cohomology of the suspension isomorphism $\tilde{H}^*(\Sigma X) \cong \tilde{H}^{*-1}(X)$.

10. Show that direct limit is an *exact functor*: that is, if $\{0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0\}_{\alpha \in I}$ is a directed system of short exact sequences of modules, then the corresponding sequence of direct limits is also exact. Use this to prove the corresponding result for long exact sequences.

11. If $\{C_*^\alpha\}_{\alpha\in I}$ is a directed system of chain complexes show that "taking homology commutes with direct limits": that is, for any n,

$$\varinjlim H_n(C_*^\alpha) = H_n(\varinjlim C_*^\alpha).$$

12. If $\{0 \to A_*^\alpha \to B_*^\alpha \to C_*^\alpha \to 0\}_{\alpha\in I}$ is a directed system of short exact sequences of chain complexes, show that "direct limit commutes with taking the associated long exact sequence".