2. The fundamental group and group actions

The fundamental group of the circle

1. Prove that S^2 minus two points is homotopy-equivalent to S^1, and that S^3 minus two points is homotopy-equivalent to S^2. Show that S^2 and S^3 are not homeomorphic.

2. By choosing suitable explicit identifications of the fundamental group of S^1 with $\mathbb{Z}$, identify the induced maps f_* of the following maps $f : S^1 \to S^1$, as maps $\mathbb{Z} \to \mathbb{Z}$:

(a). The map $z \mapsto z^n$;

(b). The antipodal map;

(c). The map $e^{i\theta} \mapsto e^{2\pi i \sin\theta}$.

3. A map $f : S^1 \to \mathbb{C} - \{0\}$ is given by the formula $z \mapsto 8z^4 + 4z^3 + 2z^2 + z^{-1}$. What is the winding number of this map about the origin, and why?

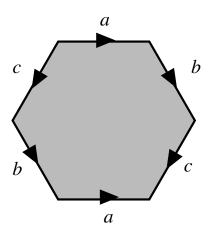
4. Identify the fundamental group of the Möbius strip M, based at the point $[(1, \frac{1}{2})]$. What class does the boundary curve of M represent in this group?

5. Recall that any path $\gamma : x \to y$ in a space X induces an isomorphism $\gamma_\# : \pi_1(X, x) \to \pi_1(X, y)$. If X is the torus $S^1 \times S^1$ and γ is a *loop* based at some point x_0, identify the induced map $\gamma_\#$.

Group actions

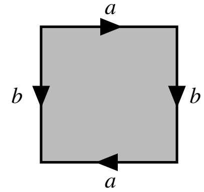
6. Define the torus T as the quotient of $\mathbb{R}^2$ by the action of the integral translation group $\mathbb{Z}^2$.

(a) Show that the space X – a solid hexagon with sides glued in pairs according to the labelled arrows shown below – is homeomorphic to T. (You don't have to write down explicit formulae; a clear description of the homeomorphism is enough.)



(b) Let $x_0 \in T$ be the image of $0 \in \mathbb{R}^2$. Show that there is an order 6 homeomorphism $f : T \to T$ which fixes x_0. With respect to the standard identification $\pi_1(T, x_0) \cong \mathbb{Z}^2$, describe the induced map $f_* : \pi_1(T, x_0) \to \pi_1(T, x_0)$ as a 2×2 integer matrix, and comment on its determinant.

7. The *Klein bottle* K is formed from a square by the following identifications.



(a). Construct a covering map from $\mathbb{R}^2$ to K, and use it to show that $\pi_1(K)$ is isomorphic to a group whose elements are pairs (m, n) of integers, with group operation given by $(m, n) * (p, q) = (m + (-1)^n p, n + q)$.

(b) Show that this group is torsion-free (contains no elements of finite order).

(c). Show that the torus is a double (2-sheeted) cover of the Klein bottle.

8. For each of the following three actions of the group $\mathbb{Z}_2$ on a sphere, compute the fundamental group of the quotient:

(a) on S^1, $z \mapsto -z$.

(b) on S^2, $(x, y, z) \mapsto (-x, -y, z)$

(c) on S^3, $(z, w) \mapsto (-z, -w)$.

Applications

9. Consider the inclusion map $i : \mathbb{R}P^2 \to \mathbb{R}P^3$ induced by including S^2 as the equator in S^3 and taking the quotients by the antipodal maps. Show that i is not homotopic to a constant map.

10. (a). Suppose there is a map $f : S^2 \to S^1$ which commutes with the antipodal maps, that is satisfies $f(-x) = -x$. Show that f induces a map $g : \mathbb{R}P^2 \to \mathbb{R}P^1$ between the quotients by the respective antipodal maps. Now, by considering a great circle path in S^2 running between the poles, derive a contradiction, showing that no such f may exist.

(b). Use this result to prove *Borsuk's theorem*, that for any (continuous) $f : S^2 \to \mathbb{R}^2$, there is a point $x \in S^2$ such that $f(x) = f(-x)$.

(c). Using Borsuk's theorem, prove the *Ham Sandwich theorem*: that there exists a plane simultaneously bisecting any three bounded measurable sets in $\mathbb{R}^3$, such as two pieces of bread and a piece of ham! (Hint: given any point on the sphere (i.e. a unit vector), consider a plane perpendicular to the vector, bisecting the ham. Take the measure theory for granted!)

11. A *topological group* is a group G whose underlying set has a topology with respect to which multiplication map $m : G \times G \to G$ and inversion $i : G \to G$ are continuous. (Remark: if the underlying topological space is actually a smooth manifold and the maps m and i are smooth, G is called a *Lie group*.) Let 1 be the identity element in the group. Prove that the fundamental group $\pi_1(G, 1)$ of a topological group is abelian. Deduce that the Klein bottle K cannot be a topological group.