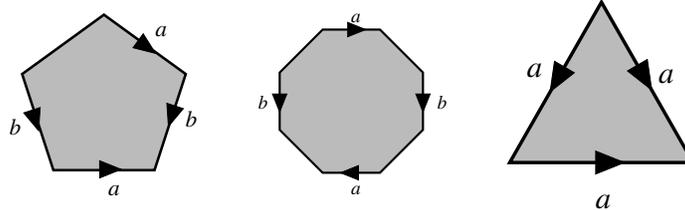


3. Van Kampen's theorem

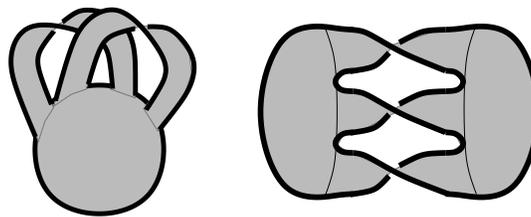
1. Recall from sheet 2 that the torus T can be made from a solid hexagon with sides glued in pairs according to the labelled arrows. Use van Kampen's theorem to compute a presentation of the fundamental group, and show that it is isomorphic to \mathbb{Z}^2 .

2. Each of the following pictures represents a solid polygon with its boundaries identified (not necessarily in pairs) according to the arrows shown. Describe the fundamental group in each case.

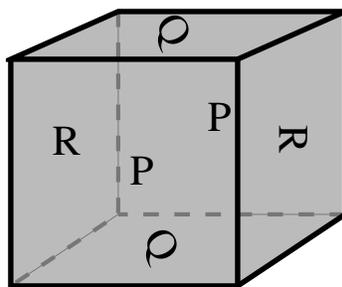


3. Let X be a torus minus a hole. Explain why X is homotopy-equivalent to $S^1 \vee S^1$. Use the fundamental group to show that there does not exist a retraction $r : X \rightarrow \partial X$.

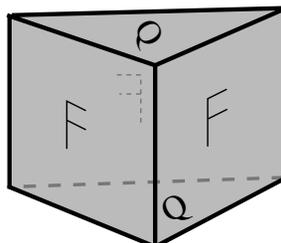
4. Each figure depicts a space made by gluing bands (copies of $I \times I$) to disc(s). For each one, compute its fundamental group, and the fundamental group of the space made by gluing a disc to each of its boundary circles.



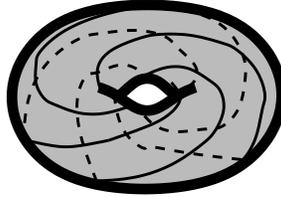
5. Take a solid cube and identify its faces in pairs, as shown below. (The top and sides are identified using translation, but the left and right faces get rotated as well as translated.) Compute the fundamental group of this space.



6. Compute the fundamental group of the space obtained by identifying the faces of a solid prism as shown below.



7. The three-sphere $S^3 = \{|z|^2 + |w|^2 = 1\}$ may be written as a union of the pieces $H_1 = \{|z|^2 \leq \frac{1}{2}\}$ and $H_2 = \{|w|^2 \leq \frac{1}{2}\}$. Check that each of these is a solid torus, homeomorphic to $B^2 \times S^1$, and that their intersection is a torus T . A picture of this decomposition is as follows (remember there's a point at infinity!):



In the picture I have also drawn a *torus knot of type* (p, q) ; it is a circle K lying in the torus T , travelling p times around longitudinally and q times meridionally, where (p, q) is a pair of coprime positive integers. By partitioning the *knot complement* $S^3 - K$ using $H_1 - K$ and $H_2 - K$, calculate a presentation for the fundamental group $\pi_1(S^3 - K)$.

8. Recall from sheet 2 the description of the Klein bottle K as a square with sides glued. It may also be built by identifying the boundary circles of two Möbius strips. Using van Kampen, calculate the presentation of $\pi_1(K)$ arising from each description. Now prove that the two groups given by these presentations are isomorphic to one another, and to the group from sheet 2 (the set of pairs (m, n) of integers, with group operation given by $(m, n) * (p, q) = (m + (-1)^n p, n + q)$).

9. Explain how to construct, given a presentation $G = \langle A : R \rangle$ of a group, a *CW-complex* whose fundamental group is G .

10. Write down the fundamental group of each of the spaces below. (No argument is needed; just name the groups – e.g. cyclic group \mathbb{Z}_n , free abelian group \mathbb{Z}^n , free group F_n , etc.).

- (a). $\mathbb{R}^2 - \{0, 1\}$
- (b). $\mathbb{R}^2 - [0, 1]$
- (c). The symbol \bigoplus (a subspace of \mathbb{R}^2)
- (d). S^2 minus four distinct points, say $\{(\pm 1, \pm 1, 0)\}$
- (e). The torus minus one point.
- (f). S^2/\mathbb{Z}_2 where the group acts via the antipodal map
- (g). S^2/\mathbb{Z}_3 where the group acts by $2\pi/3$ rotation about the z -axis
- (h). $S^2 \cup \{(0, 0, z) : -1 \leq z \leq 1\}$
- (i). $\mathbb{R}^3 - \{(x, y, 0) : x^2 + y^2 = 1\}$
- (j). $\mathbb{R}^3 - H$, where H is the *Hopf link* (a subspace consisting of two circles) shown below.



11. (This joke was invented by Zagier and others.) The *homophony group* is the group generated by the 26 letters of the alphabet, with relations $w_1 = w_2$ if these are two English words which are pronounced the same (e.g. *great = grate*). Prove that the group is trivial! (Ça marche aussi en Français!)