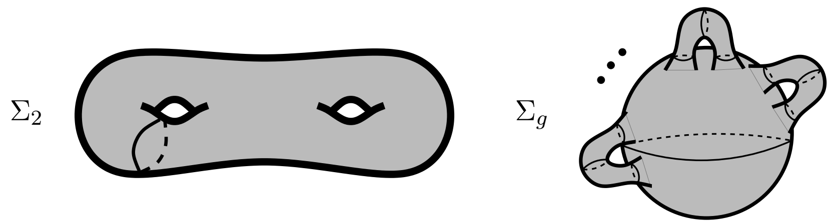
## 4. Covering spaces

### Basic theory

**1.** Show that neither the torus nor $\mathbb{R}P^2$ can cover the other.

**2.** The *closed orientable surfaces* $\Sigma_g$ *of genus* $g$ are drawn below. Construct a 5-sheeted cover $\Sigma_{11} \to \Sigma_3$ and a 3-sheeted cover $\Sigma_4 \to \Sigma_2$, and find their groups of deck transformations.




**3.** Show that the torus is a double (2-sheeted) cover of the Klein bottle. Can it be a triple cover?

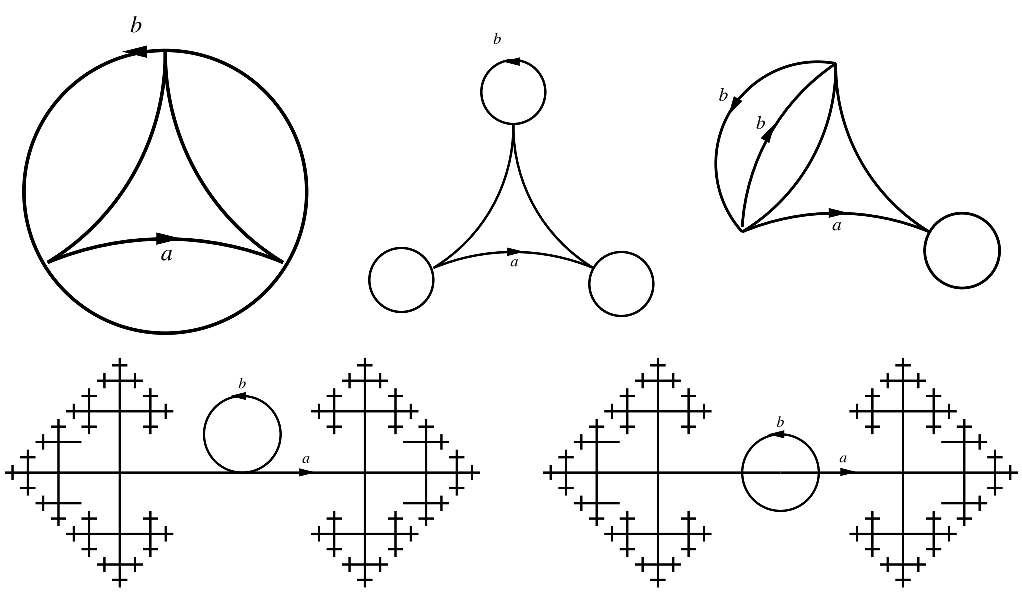
**4.** Let $G$ be a (path-connected, locally path-connected) topological group with multiplication and inversion maps $m, i$. Using existence and uniqueness of liftings, show that any path-connected cover $\tilde{G}$ of $G$ has a unique (modulo choice of basepoint living over the identity element of $G$) topological group structure such that $\tilde{G} \to G$ is a homomorphism.

**5.** Consider the group $\mathbb{Z}^n$ acting on $\mathbb{R}^n$ by integer translations, so that the quotient is the $n$-torus $T^n$. Use this covering space to prove that the homotopy groups $\pi_i(T^n)$ vanish when $i \geq 2$.

### Constructing covers

**6.** Consider the CW-complex $\Theta$ made by taking two vertices and attaching three edges between them so as to form the shape of the letter theta. Draw the the universal cover of $\Theta$ and justify your answer.

**7.** For each of the following pictures of a cover of $S^1 \vee S^1$, write down generators for their corresponding subgroups in the free group $\langle a, b \rangle$.



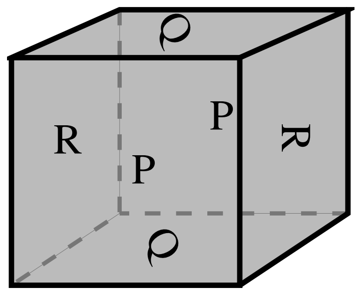
Conversely, for each of the following subgroups of $< a, b >$, draw the corresponding cover:
$\langle a^2, b^2, (ab)^2 \rangle$ $\langle a^2, b^2, aba^{-1}, bab^{-1} \rangle$ $\langle a, b^3 \rangle$ $\langle \{a^m b^n a^{-m} b^{-n} : \forall m, n\} \rangle$ $\langle \{a^{2m} b^{2n} a^{-2m} b^{-2n} : \forall m, n\} \rangle$.

**8.** Identify the fundamental group of the figure-of-eight space $S^1 \vee S^1$ with the free group $F_2 = \langle a, b\rangle$ in the usual way. Draw (and describe) the covering space corresponding to the subgroup $H$ which is generated by all elements of the form $xyx^{-1}y^{-1}$, for $x, y \in F_2$.

**9.** Draw pictures of the covers of the torus corresponding to the following types of subgroups of $\mathbb{Z}^2$, and find their groups of covering transformations (here $p, q, r, s$ are integers).

$$\langle (p,0)\rangle \qquad \langle (p,0),(0,q)\rangle \qquad \langle (p,q)\rangle \qquad \langle (p,q),(r,s)\rangle$$

**10.** Recall the "identified cube" space from the previous problem sheet. Describe its universal cover, and deduce that the space is a 3-manifold.




**Further theory**

**11.** Consider the homomorphism $\theta$ from the free group $F_2 = \langle a, b\rangle$ to $\mathbb{Z}_3$ given by sending both $a$ and $b$ to 1, and let $H$ be the kernel of $\theta$. By constructing a suitable covering space, show that $H$ is isomorphic to a free group on 4 generators. Suppose $\mathbb{Z}_3$ is replaced by $\mathbb{Z}_n$; what is the rank of $H$?

**12.** Identify $\pi_1(S^1 \vee S^1) = \langle a, b\rangle$ in the usual way. Construct the covering space corresponding to the subgroup $H$ of $F_2$ which is the kernel of the map $F_2 \to \mathbb{Z}_3$ given by sending $a \mapsto 1$ and $b \mapsto 2$.

**13.** Identify $\pi_1(S^1 \vee S^1) = \langle a, b\rangle$ in the usual way. Construct the covering space corresponding to the subgroup $H$ of the symmetric group $S_3$ which is the kernel of the map $F_2 \to S_3$ given by sending $a \mapsto (12)$ and $b \mapsto (123)$.

**14.** Identify the fundamental group of the figure-of-eight space $S^1 \vee S^1$ with the free group $F_2 =< a, b >$ in the usual way. Consider the homomorphism $\theta : F_2 \to \mathbb{Z}_4$ given by sending both $a$ and $b$ to 1, and let $H$ be the kernel of $\theta$. By constructing the covering space corresponding to $H$, show that $H$ is isomorphic to a free group on 5 generators.

**15.** A path-connected, locally path-connected, semi-locally simply connected space $X$ has fundamental group isomorphic to the symmetric group $S_3$. Up to the appropriate kind of equivalence, how many (a) based path-connected covers (b) unbased path-connected coverings (c) 3-sheeted but not necessarily path-connected coverings does $X$ have?

**16.** Say whether each of the following statements is true or false. (You don't need to show any explanation, just write "T" or "F".)

(a). Every covering space of $S^1$ is homeomorphic to a disjoint union of circles.

(b). The Klein bottle is a covering space of the torus.

(c). Each of the free groups $F_2$, $F_3$ has a subgroup isomorphic to the other.

(d). If $H$ is a finite index subgroup of the free group $F_n$, then $H$ is a free group of finite rank.

(e). The universal cover of any path-connected, locally path-connected, semi-locally simply-connected space is contractible.