1. Homotopy and the properties of the fundamental group

Homotopy

1. Show that any non-surjective map \( f : X \to S^n \) is homotopic to a constant map.

2. Let \( f, g : X \to S^n \) be such that for any \( x \in X \), \( f(x) \) and \( g(x) \) are not antipodal points on the sphere. Show that \( f \simeq g \).

3. Show that when \( n \) is odd, the antipodal map \( S^n \to S^n \), given by negation of unit vectors \( x \mapsto -x \), is homotopic to the identity map of \( S^n \).

4. A space which is homotopy-equivalent to a point is called contractible. Show that a space is contractible if and only if its identity map is homotopic to a constant map.

5. The Möbius strip \( M \) is defined as \( I \times I \) quotiented by the relation \( (x, 0) \sim (1 - x, 1), \forall x \in I \). Prove that \( S^1 \times I \) is homotopy-equivalent to the Möbius strip \( M \).

6. Show that \( \mathbb{R}^3 - S^1 \) (the complement of the unit circle in the \((x, y)\)-plane) is homotopy-equivalent to the one-point union (obtained by identifying one point from each) \( S^1 \vee S^2 \).

7. Classify the capital letters of the alphabet up to homeomorphism and up to homotopy-equivalence! (Assume that \( S^1, S^1 \vee S^1 \) and a point are not homotopy-equivalent to one another.)

8. (Tricky but important!) Let \( f, g : S^1 \to X \) be two maps from the circle to a topological space \( X \). Define a space \( P = X \cup_f B^2 \) by “attaching a disc along \( f \)”: form the disjoint union \( X \sqcup B^2 \) and then identify each point \( x \in S^1 = \partial B^2 \) with its image \( f(x) \in X \). Define \( Q = X \cup_g B^2 \) similarly. Prove that if \( f \simeq g \), then \( P \simeq Q \); thus, “the homotopy type of \( X \cup_f B^2 \) depends only on the homotopy class of the attaching map”.

Properties of the fundamental group

9. Let \( X \) be a path-connected, simply-connected (having trivial fundamental group) space, and let \( x, y \) be points of \( X \). Show that all paths from \( x \) to \( y \) are homotopic rel \( \{0, 1\} \).

10. Let \( X \) and \( Y \) be topological spaces, let \( A \) a subspace of \( X \) and let \( f : A \to Y \) be a map. A map \( F : X \to Y \) is said to be an extension of \( f \) if its restriction to \( A \) is given by \( f \). Show that the fundamental group of a path-connected space \( X \) is trivial if and only if every continuous map \( f : S^1 \to X \) has an extension to a continuous map \( F : B^2 \to X \).

11. Show that \( \pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \times \pi_1(Y, y_0) \).

12. Let \( N \) and \( S \) be the poles of the sphere \( S^n \). Supposing that \( n \geq 2 \), prove that any path in \( S^n \) may be written as a composite of finitely many paths, each of which is contained in \( S^n - \{N\} \) or \( S^n - \{S\} \), and consequently that \( \pi_1(S^n) = 1 \) for \( n \geq 2 \).