Question 1. Determine the numerical values of the following binomial coefficients as rational numbers $\frac{4}{5}$:

$$\binom{6}{3} \quad \binom{1/2}{2} \quad \binom{-2/3}{3}.$$

Solutions. The answers are $20$, $-\frac{1}{8}$, $-\frac{40}{81}$. 

Question 2. Determine the value of

$$\sum_{n=1}^{\infty} \frac{1}{n^2 + n}.$$  

Solution. This is a tough question if you don’t see the quick way to do it. Here is the hard way to do it. Notice that

$$\int_0^1 \int_0^t u^{n-1} \, du \, dt = \frac{1}{n^2 + n}.$$ 

Therefore

$$\sum_{n=1}^{\infty} = \int_0^1 \int_0^t \sum_{n=1}^{\infty} u^{n-1} \, du \, dt = \int_0^1 \int_0^t \frac{1}{1-u} \, du \, dt = \int_0^1 -\ln(1-t) \, dt = 1.$$ 

The quick way is as follows: first we see

$$\frac{1}{n^2 + n} = \frac{1}{n} - \frac{1}{n+1}.$$ 

Then the sum is

$$\sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right) = 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots$$ 

so all the terms cancel except the first one (it is a telescoping series). So the sum is 1.
**Question 3.** Find the number of compositions of $n$ in which each part is odd.

**Solution.** These are Fibonacci numbers. To see this, note that each composition $(x_1, x_2, \ldots, x_k)$ is an element of $S_1 \times S_2 \times \cdots \times S_k$ where $S_i = \{1, 3, 5, \ldots \}$ for all $i$. Therefore

$$\Phi_{S_i}(x) = \frac{x}{1 - x^2}.$$ 

By the product lemma, if $S^k = S_1 \times S_2 \times \cdots \times S_k$ then

$$\Phi_{S^k}(x) = \left( \frac{x}{1 - x^2} \right)^k.$$ 

By the sum lemma, if $S = \bigcup_{k=0}^{\infty} S^k$ then

$$\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{S^k}(x) = \sum_{k=0}^{\infty} \left( \frac{x}{1 - x^2} \right)^k = \frac{1}{1 - x/(1 - x^2)} = \frac{1 - x^2}{1 - x - x^2}.$$ 

We recognize this as the generating function for the Fibonacci numbers $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 3, a_4 = 5, a_5 = 8$ and so on.
Question 4. Prove that the generating function for the set of binary strings not containing 100 is

\[ \Phi(x) = \frac{1}{1 - 2x + x^3}. \]

Solution. Recall \( \{0\}^*\{(1\{1\}^*\{0\}{0}^*\{1\}^*)\{1\}^* \) uniquely creates all binary strings. Therefore

\[ S = \{0\}^*\{(1\{1\}^*\{0\}^*\{1\}^*)\{1\}^* \]

uniquely creates the binary strings not containing 100. The generating function is

\[ \Phi_S(x) = \frac{1}{(1 - x)^2} \cdot \frac{1}{1 - \frac{x^2}{1-x}} = \frac{1}{1 - 2x + x^3}. \]