1.9 Exercises

Question 1. Determine the number of sequences $(x_1, x_2, \ldots, x_k)$ where $x_i \in [n]$ with the given restrictions.

(a) For $i \leq n$, x_i is odd.
(b) For $i \leq n$, $x_i \neq x_{i+1}$.
(c) For $i \leq n$, $x_{i+1} > x_i$.
(d) For $i \leq n$, $x_{i+1} > x_i + 1$.

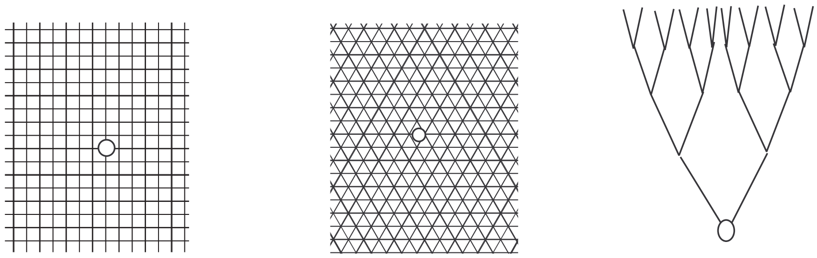

Figure 4 : Square and triangular lattices, and infinite binary tree.

Question 2. Consider the portions of the infinite lattices shown above.

(a) Determine the number of walks of length k which start at the circled point.
(b) Determine the number of walks of length k which start at the circled point but which never traverse the same edge twice in consecutive steps.
(c)* Determine the number of walks of length k which start and end at the circled point.

Question 3. Determine the number of positive integers in [1000] which are divisible by 2 or by 3 or by 11.

Question 4. Prove the inclusion-exclusion formula (Principle 3) by induction on n.

Question 5. Let $\mathbb{R}$ denote the set of real numbers, and $(0, 1)$ the set of real numbers strictly between zero and one. Find a bijection $g : (0, 1) \to \mathbb{R}$.

Question 6. Let A be the set of all sequences of positive integers of length k which add up to n, and let B be the set of all subsets of $[n-1]$ of size k. Find a bijection $f : A \to B$. Deduce that $|A| = \binom{n-1}{k-1}$ for $n \geq k \geq 1$.

Question 7. Let A be the set of all sequences of positive integers (of any length) which add up to n, and let B be the set of all subsets of $[n-1]$. Find a bijection $f : A \to B$. Deduce that $|A| = 2^{n-1}$ for $n \geq 1$.

Question 8. Prove that the number of sequences of length n with entries from $[n]$ and at least one repeated entry is

$$-\sum_{i=1}^{n-1} (-1)^{n-i} \binom{n}{i} i^n.$$

Question 9. Prove the following combinatorial identity

$$\sum_{i=k}^{n} \binom{i}{k} = \binom{n+1}{k+1}.$$

Question 10. Prove the following identity by induction on n:

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2.$$

Question 11. Prove the following inequality by induction on n:

$$\binom{2n}{n} < 4^n.$$

Question 12. Prove, by induction on k, that the number of sequences of 1s and 2s of length k which add up to n is

$$\binom{k}{n-k}.$$

Question 13. Let X be a set of size kn, where k and n are positive integers. An unordered partition of X into k-sets is a set $\{X_1, X_2, \dots, X_n\}$ such that $X_i \subseteq X$ is a set of size k, the sets X_i are disjoint, and the union of the X_i is X. Determine the number of unordered partitions of X into k-sets. For example, if $k = n = 2$, then he unordered partitions are

$$\{12, 34\} \quad \{13, 24\} \quad \{14, 23\}$$

where ij denotes the set $\{i, j\}$.

Question 14* Let X be an n-element set, where $n \geq 2$. Suppose we choose some subsets of X such that no two chosen subsets share two points and each chosen subset has at least two elements. Prove that there are distinct elements $x, y \in X$ which are in the same number of chosen subsets. For example, in the Figure 5, we have $|X| = n = 15$ (dots denote elements of X), ten chosen subsets (seven of size two, one of size three, one of size four, one of size five), and we notice that the leftmost and rightmost dots are each in three of the chosen sets.

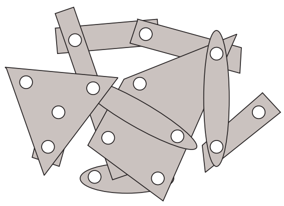

Figure 5 : Chosen subsets in a fifteen-element set.