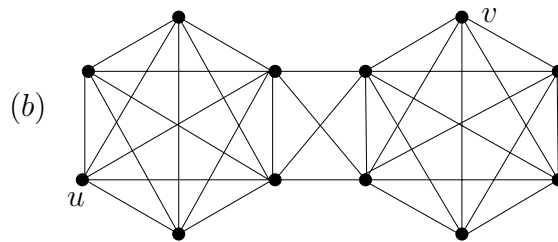
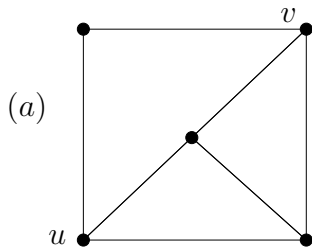


## 5.9 Exercises

**Question 1.** Determine the minimum degree  $\delta(G)$ , edge connectivity  $\lambda(G)$  and vertex connectivity  $\kappa(G)$  for each of the following two graphs. You do not need to justify your answers. Also determine  $\kappa(u, v)$  and  $\lambda(u, v)$  for the given pair  $u, v \in V(G)$ .



**Question 2.** Find an example of a graph with  $\delta(G) = d$ ,  $\kappa(G) = k$  and  $\lambda(G) = \ell$  where  $k \leq \ell \leq d$  and  $k, \ell, d$  are positive integers.

**Question 3.** Prove that if a graph has two edge-disjoint spanning trees, then that graph is 2-edge-connected.

**Question 4.** Show that every 3-connected graph has a cycle of even length.

**Question 5.** Let  $G$  be a graph with  $n \geq k + 2$  vertices such that  $\delta(G) \geq \frac{n+k-2}{2}$ . Prove that  $G$  is  $k$ -connected.

**Question 6.** Let  $G$  be a graph of maximum degree at most three. Prove that  $\lambda(G) = \kappa(G)$ . [Hint: Menger's Theorem.]

**Question 7\*** Let  $G$  be a 2-connected graph with  $m$  edges and  $n$  vertices. Prove that  $G$  contains at least  $m - n + 2$  distinct paths connecting  $u$  and  $v$  for any pair of distinct vertices  $u, v \in V(G)$ , and at least  $\binom{m-n+2}{2}$  distinct cycles. [Hint: prove this by induction on  $m$ .]

**Question 8.** Prove that every  $2n \times 2n$  bipartite graph of minimum degree at least  $n$  has a perfect matching.

**Question 9.** Let  $G(A \cup B, E)$  be a bipartite graph such that  $|B| = k|A|$ , and suppose  $|\Gamma(X)| \geq k|X|$  for each set  $X \subseteq A$ . Prove that  $G$  has a spanning subgraph in which every vertex of  $A$  has degree  $k$  and every vertex of  $B$  has degree 1.

**Question 10.** Find a maximum  $st$ -flow in the network shown in Figure 5, starting with the given flow  $f$  consisting of unit flow in the  $st$ -path of length four at the top of the diagram. Also find a minimum cut in the network. The capacities of the arcs are denoted by numbers next to each arc.

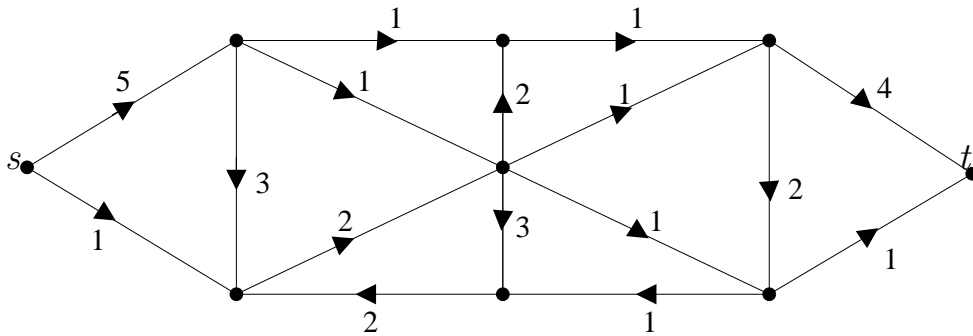


Figure 5

**Question 11.** Find a maximum  $st$ -flow in the network shown in Figure 6, starting with the zero flow. Also find a minimum cut in the network. The capacities of the arcs are shown as numbers next to each arc.

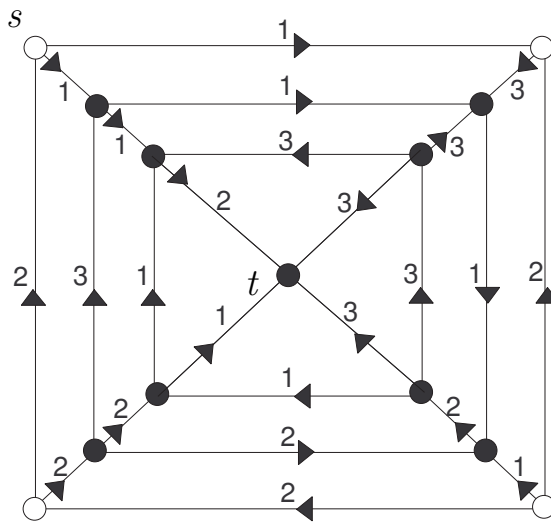


Figure 6

**Question 12.** Find a maximum  $uv$ -flow in the network shown in Figure 6, starting with the zero flow. Also find a minimum cut in the network. The capacities of the arcs are denoted by bars next to each arc.

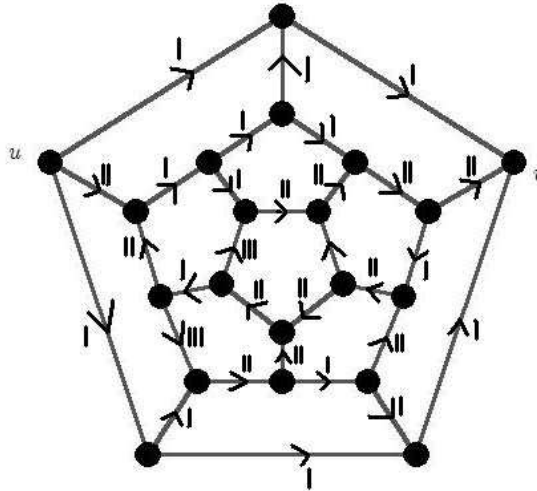


Figure 7

**Question 13\*** Find the value of  $\kappa(u, v)$  for the graph shown below. [Hint: use the proof of the vertex form of Menger's Theorem.]

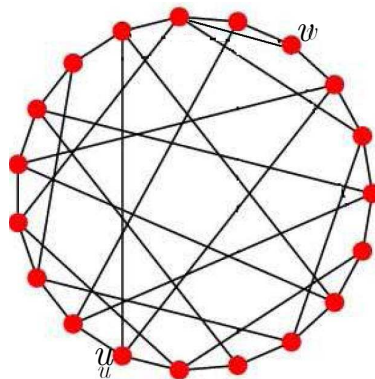


Figure 8

**Question 14.** In a network with a set  $\Sigma$  of sources and a set  $T$  of sinks, explain how you would find a maximum flow and minimum cut from  $\Sigma$  to  $T$ . [Hint: add a new source and a new sink.]

**Question 15.** Explain how to use the Max-Flow Min-Cut Theorem to find a maximum matching in a bipartite graph  $G = (A \cup B, E)$ . [Hint: proceed as in the proof of Hall's Theorem.]