

Final Examination

Math 154 – Combinatorics and Graph Theory

Instructor – J. Verstraete

Allotted time – 3 hours

Answers are to be written clearly and legibly
State clearly any theorems used without proof
Calculators are permitted
Total 50 points

Student Name

Student ID Number

Question 1.

10

Let a_n be the number of compositions of a positive integer n into any number of parts such that each part is an odd positive integer.

- (a) List the compositions of n for $n = 4$ and $n = 5$. [3]
(b) Show that the generating function for a_n is [3]

$$\Phi(x) = \frac{1 - x^2}{1 - x - x^2}.$$

- (c) Find a formula for a_n for all n . [4]

Solutions.

- (a) $(1, 1, 1, 1), (1, 3), (3, 1)$ and $(1, 1, 1, 1, 1), (1, 1, 3), (1, 3, 1), (3, 1, 1), (5)$ for $n = 4, 5$.
(b) Let $S = \bigcup_{k=0}^{\infty} S(k)$ where $S(k) = S_1 \times S_2 \times \cdots \times S_k$ and S_i is the set of odd positive integers for all i . Then

$$\Phi_{S_i}(x) = x + x^3 + x^5 + \cdots = \frac{x}{1 - x^2}$$

since it is a geometric series. By the product lemma

$$\Phi_{S(k)}(x) = \prod_{i=1}^k \Phi_{S_i}(x) = \left(\frac{x}{1 - x^2} \right)^k.$$

By the sum lemma,

$$\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{S(k)}(x) = \sum_{k=0}^{\infty} \left(\frac{x}{1 - x^2} \right)^k.$$

This is a geometric series so

$$\Phi_S(x) = \frac{1}{1 - \frac{x}{1 - x^2}} = \frac{1 - x^2}{1 - x - x^2}.$$

- (c) A recurrence for a_n from (b) is $a_n = a_{n-1} + a_{n-2}$ for $n \geq 3$ and $a_1 = a_2 = 1$. The characteristic equation is $x^2 - x - 1 = 0$ and so $x = (1 \pm \sqrt{5})/2$. Let φ and $\bar{\varphi}$ be those roots. Therefore

$$a_n = c_1 \varphi^n + c_2 \bar{\varphi}^n$$

for some constants c_1 and c_2 . Since $a_1 = a_2 = 1$, we get

$$c_1 \varphi + c_2 \bar{\varphi} = 1 \quad c_1 \varphi^2 + c_2 \bar{\varphi}^2 = 1.$$

Solving we get $c_1 = 1/\sqrt{5}$ and $c_2 = -1/\sqrt{5}$ so

$$a_n = \frac{1}{\sqrt{5}} \varphi^n - \frac{1}{\sqrt{5}} \bar{\varphi}^n.$$

Question 2.

10

- (a) Let S be the set of all binary strings starting with a 1 and not containing 101. Write down all the strings of length three and length four in S . [4]
- (b) Given that all strings in S are uniquely created by the expression [3]

$$\{1\}\{1\}^*\{0\}^* \cup \{1\}\{1\}^* (\{00\}\{0\}^*\{1\}\{1\}^*) (\{00\}\{0\}^*\{1\}\{1\}^*)^* \{0\}^*$$

show that the generating function for S is

$$\Phi_S(x) = \frac{x}{1 - 2x + x^2 - x^3}.$$

- (c) Find a recurrence equation for a_n , the number of strings of length n in S , and then find a_7 . [3]

Solutions.

- (a) They are 111, 110, 100 for length three and 1111, 1110, 1100, 1000, 1001 for length five.
- (b) Note that

$$\begin{aligned}\Phi_S(x) &= \frac{x}{(1-x)^2} + \frac{x}{(1-x)^2} \left(\frac{x^3}{(1-x)^2} \cdot \frac{1}{1 - \frac{x^3}{(1-x)^2}} \right) \\ &= \frac{x}{(1-x)^2} \left(1 + \frac{x^3}{(1-x)^2 - x^3} \right) \\ &= \frac{x}{(1-x)^2} \left(\frac{(1-x)^2 - x^3 + x^3}{1 - 2x + x^2 - x^3} \right) \\ &= \frac{x}{1 - 2x + x^2 - x^3}.\end{aligned}$$

- (c) From (b) we get $a_n = 2a_{n-1} - a_{n-2} + a_{n-3}$. From (a) we have $a_3 = 3$ and $a_4 = 5$. Also $a_2 = 2$ since 10, 11 are the only valid strings. Therefore

$$a_5 = 2a_4 - a_3 + a_2 = 10 - 3 + 2 = 9.$$

Next we get

$$a_6 = 2a_5 - a_4 + a_3 = 18 - 5 + 3 = 16.$$

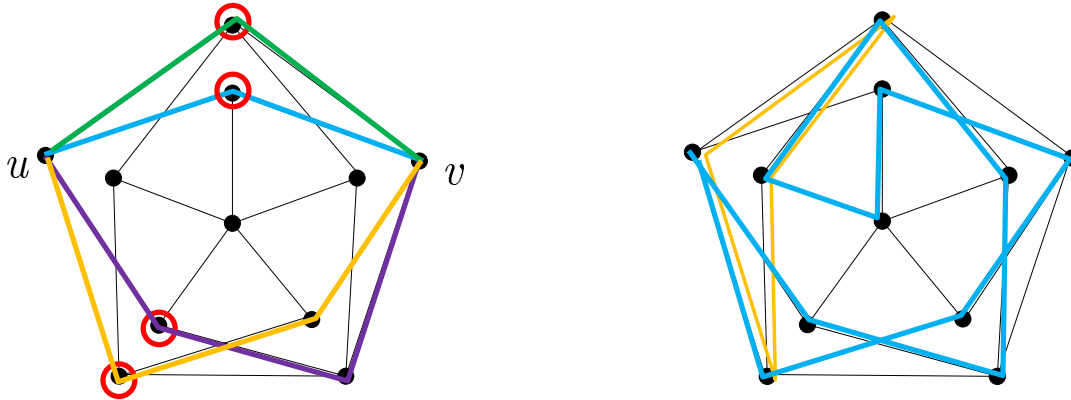
Finally

$$a_7 = 2a_6 - a_5 + a_4 = 32 - 9 + 5 = 28.$$

Question 3.

10

All parts of this question refer to the graph G shown below.



- | | |
|---|-----|
| (a) State <u>both</u> edge and vertex forms of Menger's Theorem. | [4] |
| (b) Find $\kappa(u, v), \lambda(u, v)$ for the above graph. Justify your answers. | [2] |
| (c) Find $\kappa(G), \lambda(G)$ for the above graph. Justify your answers. | [2] |
| (d) Find the lengths of the longest and the shortest cycles in G . | [2] |

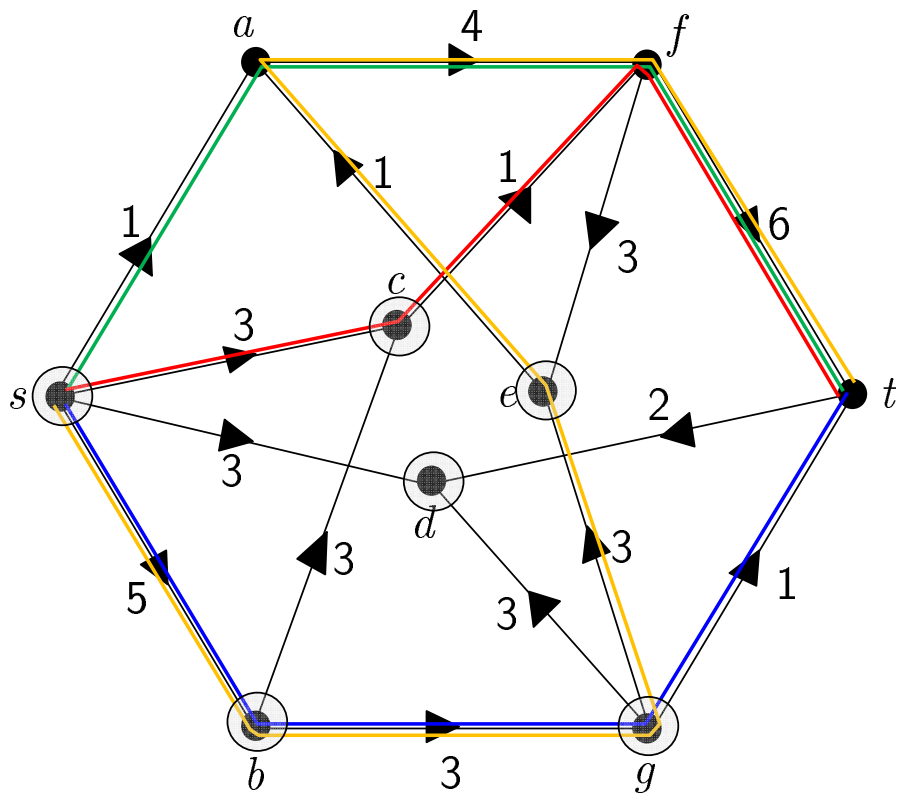
Solutions.

- (b) Since u has four neighbours (they are circled), we can separate u from v by removing those four vertices from the graph. So $\kappa(u, v) \leq 4$. Also, removing all the four edges on u , we separate u from v so $\lambda(u, v) \leq 4$. On the other hand, we can find four internally disjoint uv -paths as shown above, so $\kappa(u, v) \geq 4$, by Menger's Theorem. So $\kappa(u, v) = 4$. Since $\kappa(u, v) \leq \lambda(u, v) \leq 4$, we also get $\lambda(u, v) = 4$.
- (c) The graph has five vertices of degree three, and $\delta(G) = 3$. Since $\kappa(G) \leq \lambda(G) \leq \delta(G)$ this gives $\kappa(G) \leq \lambda(G) \leq 3$. By symmetry, if u and v are any non-adjacent vertices of G on the outer cycle of length five in G , we saw in (b) that $\kappa(u, v) = 4$. By inspection, for any other two non-adjacent vertices x and y , $\kappa(x, y) \geq 3$. Since $\kappa(G) = \min \kappa(x, y)$ we get $\kappa(G) \geq 3$ and so $\kappa(G) = 3$. It follows from $\kappa(G) \leq \lambda(G) \leq 3$ that $\lambda(G) = 3$ also.
- (d) The shortest cycle has length four (yellow) and the longest has length eleven (blue). See left figure above.

Question 5.

8

Find a maximum st -flow and minimum st -cut in the network below by applying the max-flow min-cut algorithm. Fill in the values of successive flows in the table on the next page. Then fill in a minimum cut, capacity of a minimum cut, and value of a maximum flow in the boxes in (a), (b) and (c) on the next page.



Increase flow along paths by one unit sbgt, saft, scft, sbgeaft


 = vertices in S

Figure 1 : A network

Solutions.

Increase flows along the paths shown. Fill in the table of flow values. Then the arcs in a minimum cut are

$$(S, \bar{S}) = \{(s, a), (e, a), (g, t), (c, f)\}$$

and the capacity of this cut is four. The value of a max flow is four.