2 Generating Functions

In this part of the course, we're going to introduce algebraic methods for counting and proving combinatorial identities. This is often greatly advantageous over the method of finding bijections, where it may not even be clear what each side of the identity represents. To discuss this properly, we introduce the notion of formal power series. A general idea is that if we perform algebraic or differential operations on an identity involving formal power series, then we get new and interesting identities from these operations.

2.1 Formal Power Series

We recall from analysis the notion of a power series. Whereas in analysis we are interested in convergence of power series, here we are only interested in the pattern of coefficients they represent and have little interest in convergence or not. In practical terms, this means that for us $(a_0, a_1, \dots)$ and $a_0 + a_1x + a_2x^2 + \dots$ are really the same object where $a_n \in \mathbb{Q}$ for all n, where $\mathbb{Q}$ is the field of rational numbers. For this reason, we define formal power series as follows:

Definition. *A formal power series is a series* $A(x) = a_0 + a_1x + a_2x^2 + \dots + a_kx^k + \dots$ *where* $(a_0, a_1, \dots)$ *is a finite or infinite sequence of rational numbers.*

In our work we will make no distinction between the sequence $(a_1, a_2, \dots)$ and the formal power series $A(x) = \sum_{k\geq 0} a_kx^k$. A polynomial is a formal power series whose coefficients are eventually zero – i.e. a finite formal power series. Two formal power series are equal if and only if they have the same sequence of coefficients. We write $\mathbb{Q}[[x]]$ for the set of formal power series. First we need some basic rules for adding and multiplying formal power series – in fact, formal power series behave like polynomials with respect to finite sums and products. Let's start with multiplication. If you multiply, say, the polynomials $a_0 + a_1x + a_2x^2$ and $b_0 + b_1x + b_2x^2$, you should get

$$a_0b_0 + (a_1b_0 + a_0b_1)x + (a_2b_0 + a_1b_1 + a_0b_2)x^2 + (a_2b_1 + a_1b_2)x^3 + a_2b_2x^4.$$

Notice that in front of x we see those a_ib_j such that $i + j = 1$, and in front of x^2 we see those a_ib_j such that $i + j = 2$. This generalizes: in front of x^k we expect to see a sum of a_ib_j with $i + j = k$. This gives us the rule for multiplying formal power series:

Definition. (Product rule) *Let* $A(x) = \sum a_ix^i, B = \sum b_jx^j \in \mathbb{Q}[[x]]$. *Then*

$$A(x) \cdot B(x) = \sum_{k=0}^{\infty} \left(\sum_{\ell=0}^{k} a_\ell \cdot b_{k-\ell} \right) x^k.$$

It is natural to define sums of formal power series as follows.

Definition. (Sum rule) *Let* $A(x) = \sum a_ix^i, B(x) = \sum b_jx^j \in \mathbb{Q}[[x]]$. *Then*

$$A(x) + B(x) = \sum_{k=0}^{\infty} (a_k + b_k)x^k.$$

Together with these two operations, formal power series form a set of objects on which we can do some abstract algebra, and this set of objects is called the ring of formal power series, which we also denote by $\mathbb{Q}[[x]]$. The $\mathbb{Q}$ here just means that the coefficients are rational numbers – $\mathbb{Q}$ is standard notation for the set of rational numbers.

2.1.1 The inverse

A very important notion concerning formal power series is the notion of the inverse. In the rational numbers, which we denote by $\mathbb{Q}$, the inverse of a non-zero number a is a number b such that $ab = 1$. The same definition is made for formal power series:

Definition. (Inverse) *The inverse of a formal power series A is a formal power series B such that $A \cdot B = 1 = B \cdot A$, in which case we write $B := \frac{1}{A}$ or $B := A^{-1}$.*

This definition is unambiguous, since if B_1 and B_2 are inverses of A, then $B_1 = B_1AB_2 = B_2$, so the inverse of A is unique. It is not hard to see which formal power series have inverses and which do not. If a formal power series $\sum_{k=0}^{\infty} a_k x^k$ has $a_0 = 0$, then it can't have an inverse. In fact, the following theorem is true:

Theorem 1 *A formal power series $A = \sum_{k=0}^{\infty} a_k x^k$ has an inverse if and only if $a_0 \neq 0$.*

Since x has no inverse, this means that $\frac{1}{x}$ is meaningless in this setting. However $1 - x$ has an inverse, so $\frac{1}{1-x}$ is meaningful and is a formal power series in $\mathbb{Q}[[x]]$. It is natural to ask what the coefficients are, and how in general to find the inverse of a formal power series. In this section, we will not answer this question, except in two important special cases. The first is the inverse of $1 - x$: observe that by the product rule,

$$(1-x)(1+x+x^2+\ldots) = 1 + (1\cdot 1 - 1\cdot 1)x + (1\cdot 1 - 1\cdot 1)x^2 + \ldots = 1$$

so by definition the inverse of $1 - x$ is the geometric formal power series:

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}.$$

Similarly, as one can check by multiplying things out,

$$\sum_{k=0}^{n} x^k = \frac{1-x^{n+1}}{1-x}.$$

These two formulas will be used repeatedly in later work.

2.1.2 Derivative and integral

Two very important functions on $\mathbb{Q}[[x]]$, which we denote by $\frac{d}{dx}$ and $\int$ are called formal differentiation and integration, respectively:

Definition. (Formal Derivative and Integral) *Let $A = a_0 + a_1x + a_2x^2 + \ldots \in \mathbb{Q}[[x]]$. Then the formal derivative of A is*

$$\frac{d}{dx}A = a_1 + 2a_2x + 3a_3x^2 + \ldots = \sum_{k=1}^{\infty} ka_kx^{k-1}.$$

The formal integral of $A(x)$ is

$$\int A = a_0 x + \frac{1}{2}a_1 x^2 + \frac{1}{3}a_2 x^3 + \ldots = \sum_{k=0}^{\infty} \frac{1}{k+1} a_k x^{k+1}.$$

To summarize, we integrate and differentiate formal power series term by term.

2.2 The Binomial Theorem

By Theorem 1, we know exactly which formal power series have inverses. But one would like to find the inverse of a given formal power series – in other words, to know what the sequence of coefficients is. We already saw the answer for $1-x$: the sequence of coefficients in the inverse is $(1,1,1,\ldots)$ since the inverse is $1+x+x^2+x^3+\ldots$. In this section we answer the question for powers of $(1+x)$ via the Binomial Theorem:

Theorem 2 (General Binomial Theorem) Let a be a rational number. Then

$$(1+x)^a = \sum_{k=0}^{\infty} \binom{a}{k} x^k$$

where

$$\binom{a}{k} := \frac{a(a-1)(a-2)\ldots(a-k+1)}{k!} \quad \text{and} \quad \binom{a}{0} := 1.$$

The binomial theorem agrees with the geometric power series. Since

$$\binom{-1}{k} = \frac{(-1)\cdot(-2)\cdots(-k)}{k!} = (-1)^k$$

we obtain

$$(1-x)^{-1} = \sum_{k=0}^{\infty} \binom{-1}{k} (-x)^k = \sum_{k=0}^{\infty} x^k.$$

A key point in the binomial theorem is the definition of $\binom{a}{k}$. We will consider a few special cases. When a is a positive integer n, we get

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

which agrees with the basic combinatorial version of the binomial theorem. Note that $\binom{n}{k} = 0$ if $k > n$, which is why the sum stops at n. In general, $\binom{a}{k}$ is non-zero for all k, so the binomial theorem is actually an infinite sum. Some further special cases of the binomial theorem which we encounter somewhat frequently are given in the next two theorems.

Theorem 3 *Let n be a positive integer. Then*

$$(1-x)^{-n} = \sum_{k=0}^{\infty} \binom{n+k-1}{n-1} x^k.$$

Proof ▷ First we deal with the binomial coefficient $\binom{-n}{k}$. For $k=0$, $\binom{-n}{k}=1$. By definition, for $k \geq 1$,

$$\begin{aligned}\binom{-n}{k} &= \frac{-n(-n-1)(-n-2)\dots(-n-k+1)}{k!} \\ &= \frac{(-1)^k n(n+1)(n+2)\dots(n+k-1)}{k!} \\ &= (-1)^k\frac{(n+k-1)(n+k-2)\dots n}{k!} \\ &= (-1)^k\binom{n+k-1}{k}.\end{aligned}$$

Therefore by the binomial theorem

$$(1-x)^{-n}=\sum_{k=0}^{\infty}\binom{-n}{k}(-x)^k=\sum_{k=0}^{\infty}\binom{n+k-1}{k}x^k.$$

This completes the proof. ∎

We could have also written $\binom{n+k-1}{n-1}$ instead of $\binom{n+k-1}{k}$, since these quantities are equal.

Theorem 4

$$(1+x)^{\frac{1}{2}}=1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{2^{2k-1}k}\binom{2k-2}{k-1}x^k.$$

Proof ▷ First we consider $\binom{\frac{1}{2}}{k}$. By definition, $\binom{\frac{1}{2}}{0}=1$. For $k \geq 1$,

$$\begin{aligned}\binom{\frac{1}{2}}{k} &= \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-k+1)}{k!} \\ &= \frac{1\cdot(-1)\cdot(-3)\cdot(-5)\cdots(-2k+3)}{2^k k!} \\ &= \frac{(-1)^{k-1}}{2^k}\frac{1\cdot 3\cdot 5\cdots(2k-3)}{k!} \\ &= \frac{(-1)^{k-1}}{2^k}\frac{1\cdot 2\cdot 3\cdot 4\cdots(2k-2)}{k!\cdot 2\cdot 4\cdot 6\cdot(2k-2)} \\ &= \frac{(-1)^{k-1}}{2^k}\frac{(2k-2)!}{k!\cdot 2^{k-1}\cdot(k-1)!} \\ &= \frac{(-1)^{k-1}}{k2^{2k-1}}\frac{(2k-2)!}{(k-1)!(k-1)!} \\ &= \frac{(-1)^{k-1}}{k2^{2k-1}}\binom{2k-2}{k-1}\end{aligned}$$

Therefore by the binomial theorem

$$(1+x)^{\frac{1}{2}}=\sum_{k=0}^{\infty}\binom{\frac{1}{2}}{k}x^k=1+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k2^{2k-1}}\binom{2k-2}{k-1}x^k.$$

Note that we separate the case $k=0$ from $k \geq 1$ in the proof. ∎

2.3 Combinatorial Identities Revisited

The following identities hold for positive integers n:

$$\sum_{k \text{ odd}} \binom{n}{k} = \sum_{k \text{ even}} \binom{n}{k} \qquad \sum_{k=1}^{n} k \binom{n}{k} = n2^{n-1}.$$

The Binomial Theorem gives an algebraic way to prove these identities, using formal powers series in $\mathbb{Q}[[x]]$. Consider, for example, the first identity. We proved it by finding a bijection from the set of subsets of $[n]$ of odd size to the set of subsets of $[n]$ of even size. But using the binomial theorem, we can write

$$0 = (1-1)^n = \sum_{k=0}^{n} \binom{n}{k} (-1)^k.$$

This immediately gives the identity, since the sum is exactly

$$\sum_{k \text{ odd}} \binom{n}{k} - \sum_{k \text{ even}} \binom{n}{k}.$$

Consider the second identity. If we can prove

$$\sum_{k=1}^{n} k \binom{n}{k} x^{k-1} = n(1+x)^{n-1}$$

then we are done, since we can substitute $x = 1$ (these are finite sums, so that is permitted). Well the binomial theorem says

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

Taking the formal derivative we get

$$n(1+x)^{n-1} = \sum_{k=1}^{n} k \binom{n}{k} x^{k-1}$$

as required. So proving combinatorial identities often becomes a mechanical task when one appeals to formal power series. We now present a few more examples of proving combinatorial identities using the binomial theorem.

Examples. The following identity follows immediately by formally integrating the formula in the binomial theorem, and then substituting $x = 1$:

$$\sum_{k=0}^{n} \frac{1}{k+1} \binom{n}{k} = \frac{2^{n+1}-1}{n+1}.$$

The right hand side is actually

$$\int_0^1 (1+x)^{n+1} dx = \frac{2^{n+1}-1}{n+1}.$$

When we integrate a formal power series, it corresponds to the term-by-term integral of the corresponding power series when that integral exists: we define

$$\int \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \int_0^x a_n t^n dt.$$

Here is another example: we proved earlier that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} = 3^n$$

using bijections. But it follows immediately upon setting $a = n$ and $x = 2$ in the binomial theorem.

2.4 Identities from coefficients

Let $[x^k]A(x)$ denote the coefficient of x^k in the formal power series $A(x)$. If $A(x) = \sum_{k=0}^{\infty} a_k x^k$, then this is precisely a_k. Many combinatorial identities are obtained by expanding $(1+x)^a$ in two different ways, and then comparing coefficients of the resulting formal power series. This will be the general idea in the next section, on generating functions, but we start with an example.

Suppose we want to prove the identity

$$\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$$

where $k \le n$ and m are positive integers. If we multiply the right hand side by x^k and then sum up to $n+m$, we get

$$A(x) := \sum_{k=0}^{\infty} \binom{n+m}{k} x^k = (1+x)^{n+m}$$

where we used the Binomial Theorem (actually the sum on the left is finite since all terms are zero when $k > n+m$). However $A(x) = B(x)$ where

$$B(x) = (1+x)^n(1+x)^m = \left(\sum_{i=0}^{n} \binom{n}{i} x^i\right) \cdot \left(\sum_{r=0}^{n} \binom{m}{r} x^r\right) = \sum_{i=0}^{n}\sum_{r=0}^{m} \binom{n}{i}\binom{m}{r} x^{i+r}.$$

Therefore

$$\binom{n+m}{k} = [x^k]A(x) = [x^k]B(x) = \sum_{i+r=k} \binom{n}{i}\binom{m}{r} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}.$$

This proves the identity.

One of the most famous identities involving binomial coefficients is the Pascal Triangle Identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

It is possible to prove this using bijections: for a set $A \subseteq [n]$ of size k, define

$$f(A) = \begin{cases} A\backslash[n] & \text{if } n \in A \\ A & \text{if } n \notin A \end{cases}$$

Then f maps A to either a set of size $[k]$ in $[n-1]$ – namely A – or f maps A to a set of size $[k-1]$ in $[n-1]$ – namely $A\backslash[n]$. Also, as can be checked, f is a bijection since the inverse of f is

$$f(B) = \begin{cases} B \cup [n] & \text{if } |B| = k-1 \\ B & \text{if } |B| = k \end{cases}$$

However, the identity can also be proved using the binomial theorem:

$$(1+x)^n = (1+x)(1+x)^{n-1} = (1+x)\sum_{j=0}^{n-1}\binom{n-1}{j}x^j = \sum_{k=0}^{n-1}\binom{n-1}{j}x^j + \sum_{j=0}^{n-1}\binom{n-1}{j}x^{j+1}.$$

The last sum is

$$\sum_{j=1}^{n}\binom{n-1}{j-1}x^j.$$

Comparing coefficients of x^k, we get

$$\binom{n}{k} = [x^k](1+x)^n = [x^k]\sum_{k=0}^{n-1}\binom{n-1}{j}x^j + [x^k]\sum_{j=0}^{n-1}\binom{n-1}{j}x^{j+1} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

This proves the identity.

2.5 Generating Functions

Let S be a set. A weight function on S is a function $\omega : S \to \mathbb{Z}_{\geq 0}$, where $\mathbb{Z}_{\geq 0}$ denotes the set of non-negative integers. In other words, each element of the set S has a non-negative weight attached to it. Sometimes we will refer to the elements of S as configurations, and a generic configuration will be denoted by σ, and $\omega(\sigma)$ is the weight of σ. Most problems in combinatorics can be formulated in the following way:

Determine the number of elements of S of weight k.

For example, if S is the set of all subsets of $[n]$, and the weight of a set in S is its size, then the above problem is to determine the number of subsets of $[n]$ of size k. Here the weight function is $\omega(\sigma) = |\sigma|$ for $\sigma \in S$. As another example, if S is the set of sequences of positive integers, and the weight of a sequence is the sum of the entries of the sequence, then the above problem is to determine the number of sequences (configurations) of integers which add up to k. We will approach these problems using the notion of generating functions.

Definition. Let S be a set of configurations with weight function ω. Then the generating function (or generating series) for S with respect to ω is

$$\Phi_S(x) := \sum_{\sigma \in S} x^{\omega(\sigma)}.$$

This is not the most useful form of the generating function: it is more useful to write

$$\Phi_S(x) = \sum_{k=0}^{\infty} \sum_{\substack{\sigma \in S \\ \omega(\sigma)=k}} x^{\omega(\sigma)} = \sum_{k=0}^{\infty} a_k x^k$$

where a_k is the number of configurations of weight k in S. Thus a_k is the solution to the general problem we gave above, and we can find a_k by looking at the coefficient of x^k in the generating function. Equivalently, we can find the generating function if we know every a_k. Let's consider some examples:

Example. Let $S = \mathbb{Z}_{\geq 0}$ and define $\omega(\sigma) = \sigma$ for $\sigma \in S$. Then $a_k = 1$ for every k (there is only one configuration of weight k, namely the number k), so

$$\Phi_S(x) = \sum_{k=0}^{\infty} x^k.$$

We can write this more compactly using the formula for geometric series. Recall that

$$\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$$

where, if x is a number, then $|x| < 1$. So the generating function is

$$\Phi_S(x) = \frac{1}{1-x}.$$

Example. Let $S = \mathbb{Z}_{\geq 0}$ and define $\omega(\sigma) = \sigma$ if σ is odd, and $\omega(\sigma) = \sigma/2$ if σ is even. We have to work out a_k. Well how many configurations have weight k? If k is even, then only $2k$ has weight k. If k is odd, then both $2k$ and k have weight k. So

$$\Phi_S(x) = \sum_{k \text{ odd}} 2x^k + \sum_{k \text{ even}} x^k.$$

We can write this more succinctly using the geometric series

$$\Phi_S(x) = \sum_{m=0}^{\infty} 2x^{2m+1} + \sum_{m=0}^{\infty} x^{2m}.$$

Using geometric series

$$\sum_{m=0}^{\infty} x^{2m} = \frac{1}{1-x^2}$$

so the generating function is

$$\Phi_S(x) = \frac{2x}{1-x^2} + \frac{1}{1-x^2} = \frac{2x+1}{1-x^2}.$$

Example. Let S be the set of subsets of $[n]$, where $\omega(\sigma) = |\sigma|$. Then a_k is the number of configurations (sets) of weight (size) k, so $a_k = \binom{n}{k}$ by our theorem on sets from Chapter 1. So

$$\Phi_S(x) = \sum_{k=0}^{\infty} \binom{n}{k} x^k = \sum_{k=0}^{n} \binom{n}{k}.$$

Note that we stopped the sum at $k = n$ simply because for $k > n$, $\binom{n}{k} = 0$. Now the binomial theorem applies:

$$\Phi_S(x) = (1 + x)^n$$

and what could be simpler? The generating function for subsets of $[n]$ is just $(1 + x)^n$!

Example. Let S be the set of non-empty subsets of $[n]$ where $\omega(\sigma) = \max \sigma$, the maximum element of σ. Then a_k is the number of sets whose maximum element is σ. Clearly $a_0 = 0$, $a_1 = 1$, $a_2 = 2$, but how do we find a_k? Well first we have to put k in the set. Then we can choose any subset of $[k-1]$ together with k to make the set, and there are 2^{k-1} ways to do that for $k \geq 1$. So $a_k = 2^{k-1}$ for $k \geq 1$ and $a_0 = 0$ and

$$\begin{aligned}
\Phi_S(x) &= \sum_{k=1}^{n} 2^{k-1} x^k \\
&= x \sum_{k=0}^{n-1} (2x)^k \\
&= \frac{x(1 - (2x)^n)}{1 - 2x}.
\end{aligned}$$

Here we used the finite geometric series formula (see last section on formal power series)

$$\sum_{k=0}^{n} x^k = \frac{1 - x^{n+1}}{1 - x}.$$

Example. Consider the last example with S the same, but ω defined by $\omega(\sigma) = \min \sigma$, observe that $a_0 = 0$, and $a_k = 2^{n-k}$, since each set whose minimum element is k is of the form $\{k\} \cup A$ where $A \subseteq \{k+1, k+2, \ldots, n\}$, and there are 2^{n-k} choices for A. Therefore

$$\begin{aligned}
\Phi_S(x) &= \sum_{k=1}^{n} a_k x^k \\
&= \sum_{k=1}^{n} 2^{n-k} x^k \\
&= 2^n \sum_{k=1}^{n} (x/2)^k \\
&= x 2^{n-1} \sum_{k=0}^{n-1} (x/2)^k \\
&= \frac{x 2^{n-1}(1 - (x/2)^n)}{1 - x/2} \\
&= \frac{x(2^n - x^n)}{2 - x}.
\end{aligned}$$

Again we used the finite geometric series formula.

2.6 Average Weight

Generating functions are just instances of formal power series. All the manipulations we have done with generating functions are valid, by the definitions in the last section on formal power series, for example, writing $1/(1-x)$ is meaningful. Our first theorem on generating functions allows us to work out average weight of elements of S using the derivative of the generating function. The average weight is defined by

$$\frac{\sum_{\sigma \in S} \omega(\sigma)}{|S|} = \frac{\sum_{k=0}^{\infty} k a_k}{\sum_{k=0}^{\infty} a_k}.$$

Theorem 5 *Let S be a finite set of configurations with weight function ω. Then the average weight of an element in S is $\Phi'_S(1)/\Phi_S(1)$, where $\Phi'_S(1)$ means $\frac{d}{dx}\Phi_S(x)$ evaluated at $x = 1$.*

Proof ▷ Clearly $\Phi_S(1) = \sum a_k = |S|$. Now

$$\frac{d}{dx}\Phi(x) = \sum_{k=1}^{\infty} k a_k x^{k-1}$$

so when $x = 1$ we get

$$\sum_{k=1}^{\infty} k a_k = \sum_{\sigma \in S} \omega(\sigma)$$

which is the sum of the weights of configurations. So the average weight is as stated. ▮

Example. It is intuitively clear that the average size of a subset of $[n]$ is $\frac{n}{2}$. This can be proved without generating functions, using the fact that $\binom{n}{k} = \binom{n}{n-k}$. The generating function for the set S of subsets of $[n]$ with $\omega(\sigma) = |\sigma|$ for $\sigma \in S$ is

$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$$

by the binomial theorem, since $a_k = \binom{n}{k}$ for $0 \leq k \leq n$ and $a_k = 0$ otherwise. Therefore $\Phi_S(x) = (1+x)^n$. Now $\Phi'_S(1) = n2^{n-1}$ and $\Phi_S(1) = 2^n$, so the average size of a set is $\Phi'_S(1)/\Phi_S(1) = n/2$, as required.

Example. Let's work out the average minimum element of a set in $[n]$ (excluding the empty set). The generating function is (see a preceding example)

$$\Phi_S(x) = \frac{x(2^n - x^n)}{2-x}.$$

So $\Phi_S(1) = 2^n - 1$. The derivative of Φ is

$$\frac{2^n - x^n - nx^n}{2-x} + \frac{x(2^n - x^n)}{(2-x)^2}.$$

When $x = 1$, this is $2^{n+1} - 2 - n$. So by the theorem, the average minimum element of a set in $[n]$ is

$$\frac{2^{n+1} - 2 - n}{2^n - 1} = 2 - \frac{n}{2^n - 1}.$$

So the average minimum element is less than two. It is left as an exercise to work out the average maximum element.

2.7 Sum and Product Lemma

Two of the most useful facts concerning generating functions are the analogues of the multiplication and summation principle, and use the sum and product rules of formal power series. The sum lemma is as follows:

Lemma 6 (Sum lemma) *Let S be a set of configurations, and let (A, B) be a partition of S. Then*

$$\Phi_S(x) = \Phi_A(x) + \Phi_B(x).$$

Proof $\triangleright$ Using the definition of generating functions:

$$\begin{aligned}\Phi_S(x) &= \sum_{\sigma \in S} x^{\omega(\sigma)} \\ &= \sum_{\sigma \in A} x^{\omega(\sigma)} + \sum_{\sigma \in B} x^{\omega(\sigma)} \\ &= \Phi_A(x) + \Phi_B(x).\end{aligned}$$

In the last two lines we used the sum rule for formal power series. ∎

More generally, to find the generating function of S knowing the generating functions for the parts $A_1, A_2, \ldots, A_n$ in a partition of S, we just add together the generating functions for all A_is. The product lemma is only a bit more complicated. Recall the Cartesian product of sets A and B is the set $A \times B = \{(a, b) : a \in A, b \in B\}$. If A and B are sets of configurations with weight functions ω_A and ω_B, then $A \times B$ is also a set of configurations, but we have to define a weight function. In many cases, the natural weight function is $\omega_A + \omega_B$, and it is to this situation that the product lemma applies:

Lemma 7 (Product lemma) *Let A, B be sets of configurations with weight functions ω_A and ω_B, and let $S = A \times B$ with weight function $\omega(a, b) = \omega_A(a) + \omega_B(b)$. Then*

$$\Phi_S(x) = \Phi_A(x) \cdot \Phi_B(x).$$

Proof $\triangleright$ Using the definition of generating functions:

$$\begin{aligned}\Phi_S(x) &= \sum_{\sigma \in S} x^{\omega(\sigma)} \\ &= \sum_{(a,b) \in A \times B} x^{\omega_A(a) + \omega_B(b)} \\ &= \sum_{a \in A} \sum_{b \in B} x^{\omega_A(a)} x^{\omega_B(b)} \\ &= \Phi_A(x) \cdot \Phi_B(x).\end{aligned}$$

In the last line we used the product rule of formal power series. ∎

We will spend most of the rest of our time with generating functions on applications of the product and sum lemmas to generating functions arising from important counting problems, especially to counting compositions and binary strings.

2.8 Compositions

A composition of a positive integer n is a sequence $(x_1, x_2, \ldots, x_k)$ of positive integers such that $n = x_1 + x_2 + \ldots + x_k$. The numbers x_i are called the parts of the composition. The central question of this section is to determine the number of compositions of n under various restrictions on x_i. We first consider the problem of counting compositions of n into k parts, using generating functions. Let S denote the set of all sequences $(x_1, x_2, \ldots, x_k)$ of positive integers, with weight function $\omega(x_1, x_2, \ldots, x_k) = x_1 + x_2 + \ldots + x_k$. If $S_i = \mathbb{Z}^+$ with weight function $\omega_i(x) = x$, then

$$\Phi_{S_i}(x) = x + x^2 + x^3 + \ldots = \frac{x}{1-x}$$

by definition of the generating function and the infinite geometric series formula. However $S = S_1 \times S_2 \times \ldots \times S_k$ and the definition of the weight function ω ensures that the conditions of the product lemma are met. We conclude by the product lemma that

$$\Phi_S(x) = \prod_{i=1}^{k} \Phi_{S_i}(x) = \left(\frac{x}{1-x}\right)^k.$$

By definition of $\Phi_S(x)$, the coefficient of x^n in the generating function is exactly the number of compositions of n: it is the number of configurations (i.e. sequences $(x_1, x_2, \ldots, x_k)$) of weight n (i.e. with $x_1 + x_2 + \ldots + x_k = n$, by definition of the weight function ω). So all we have to do is locate x^n in $\Phi_S(x)$. However, the formula for $\Phi_S(x)$ given above is not written as a sum of powers of x, so we have to expand it as a sum to see x^n. At this point, we recall the notation $[x^n]\Phi_S(x)$ as shorthand for "the coefficient of x^n in $\Phi_S(x)$". To expand the generating function, we use the binomial theorem:

$$\Phi_S(x) = x^k(1-x)^{-k} = x^k \sum_{j=0}^{\infty} \binom{-k}{j} (-1)^j x^j.$$

To get $[x^n]\Phi_S(x)$, we put $j = n - k$ in the sum. Then

$$[x^n]\Phi_S(x) = (-1)^{n-k} \binom{-k}{n-k}.$$

Let's recall a useful fact about binomial coefficients from the proof of Theorem 3:

Lemma 8 *Let k, j be positive integers. Then*

$$\binom{-k}{j} = (-1)^j \binom{k+j-1}{j}.$$

Therefore

$$[x^n]\Phi_S(x) = \binom{n-1}{k-1}$$

and this, finally, is the number of compositions of n into k parts. The above proof generalizes to counting other types of compositions $(x_1, x_2, \ldots, x_k)$ of n, so we list general steps to take when approaching such problems:

Procedure for counting compositions

(1) *Write the set S of sequences to be counted.*
(2) *Write S_i, the set of possible values of x_i, and observe $S_1 \times S_2 \times \ldots \times S_k$.*
(3) *Write $\Phi_{S_i}(x)$ in closed form.*
(4) *Write $\Phi_S(x) = \prod_{i=1}^k \Phi_{S_i}(x)$ using the product lemma.*
(5) *Using the binomial theorem, expand $\Phi_S(x)$ as a sum.*
(6) *Determine $[x^n]\Phi_S(x)$ by carefully choosing the summation variable in (5).*
(7) *Use Lemma 8 to write $[x^n]\Phi_S(x)$ in terms of "positive" binomial coefficients.*

2.8.1 Restrictions on compositions

First Variation. We determine the number of sequences $(x_1, x_2, \ldots, x_k)$ which add up to n, such that $x_i \geq 0$ for $i \in [k]$. Step (1) gives

$$S = \{(x_1, x_2, \ldots, x_k) : x_i \geq 0, i \in [k]\}.$$

Step (2) is to write $S_i = \mathbb{Z}^+ \cup \{0\}$, the set of non-negative integers, for $i \in [k]$. Then $S = S_1 \times S_2 \times \ldots \times S_k$. Step (3) : the generating function for S_i is

$$\Phi_{S_i}(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

Therefore in Step (4), by the product lemma, we get

$$\Phi_S(x) = \prod_{i=1}^k \Phi_{S_i}(x) = (1-x)^{-k}.$$

By the binomial theorem, Step (5),

$$\Phi_S(x) = \sum_{j=0}^{\infty} \binom{-k}{j} (-1)^j x^j.$$

In Step (6), the coefficient of x^n (which is the answer to the problem) is found by setting $j = n$, in which case we get

$$[x^n]\Phi_S(x) = \binom{-k}{n}(-1)^n.$$

By Lemma 8, in Step (7) we obtain

$$(-1)^n \binom{-k}{n} = \binom{n+k-1}{k-1}$$

In conclusion, there are $\binom{n+k-1}{k-1}$ sequences of k non-negative integers adding up to n. We can check this for a small value of n, say $n = 3$ and $k = 3$: the sequences are

$$(0,0,3)\ (3,0,0)\ (0,3,0)\ (1,1,1)\ (2,1,0)$$
$$(1,2,0)\ (1,0,2)\ (2,0,1)\ (0,1,2)\ (0,2,1)$$

There are ten sequences, which agrees with $\binom{3+3-1}{3-1} = \binom{5}{2} = 10$.

Second Variation. Suppose we want to count compositions where $x_i \geq 1$ is odd. Defining in Step (1) $S = \{(x_1, x_2, \ldots, x_k) : x_i \text{ is odd}\}$, and in Step (2), $S_i = \{1, 3, 5, 7, \ldots,\}$, we get in Step (3)

$$\Phi_{S_i}(x) = x + x^3 + x^5 + \ldots = \frac{x}{1 - x^2}$$

using the geometric series formula. In Step (4), By the product lemma

$$\Phi_S(x) = \left(\frac{x}{1 - x^2}\right)^k.$$

Appealing to the binomial theorem in Step (5),

$$\Phi_S(x) = x^k (1 - x^2)^{-k} = x^k \sum_{j=0}^{\infty} \binom{-k}{j} (-1)^j x^{2j}.$$

To obtain the coefficient of x^n in Step (6), put $j = (n - k)/2$ if $n - k$ is even. It is very important to note that if $n - k$ is odd, then the answer to the problem is zero, since j only takes on integer values. So for $n - k$ even, the answer is

$$\binom{-k}{(n - k)/2} = \binom{(n + k)/2 - 1}{k - 1}$$

where we applied Lemma 8 in Step (7). Again it is a good idea to check that this is the answer: suppose $n = 7$ and $k = 3$, then the sequences are

$$\begin{array}{c} (1, 3, 3)\ (3, 3, 1)\ (3, 1, 3) \\ (1, 1, 5)\ (1, 5, 1)\ (5, 1, 1) \end{array}$$

and so there are six sequences. This agrees with $\binom{(7+3)/2-1}{3-1} = \binom{4}{2} = 6$. The cautionary note from this example is that in Step (6), the answer may differ for different values of n, so one must take care when finding the coefficient of x^n.

Third Variation. Suppose we insist $x_i \geq i$ for all $i \in [k]$. Defining in Step (2) $S_i = \{i, i + 1, i + 2, \ldots\}$, we see that in Step (3)

$$\Phi_{S_i}(x) = \frac{x^i}{1 - x}$$

using the formula for finite geometric series. As in the last example, with $S = S_1 \times S_2 \times \ldots \times S_k$ in Step (1), Step (4) and the product lemma give

$$\Phi_S(x) = \prod_{i=1}^{k} \Phi_{S_i}(x) = \prod_{i=1}^{k} \frac{x^i}{1 - x} = (1 - x)^{-k} x^{1+2+\ldots+k}.$$

It is well known that

$$1 + 2 + \ldots + k = \binom{k + 1}{2}$$

so

$$\Phi_S(x) = x^{\binom{k+1}{2}} (1 - x)^{-k}.$$

The binomial theorem in Step (5) and Step (7) give

$$\Phi_S(x) = x^{\binom{k+1}{2}} \sum_{j=0}^{\infty} \binom{k+j-1}{k-1} x^j.$$

Therefore Step (6) gives

$$[x^n]\Phi_S(x) = \binom{k+n-\binom{k+1}{2}-1}{k-1} = \binom{n-\binom{k}{2}-1}{k-1}$$

which is valid for $n \geq \binom{k}{2}$. This answers the problem, and can be checked against some small values of n and k. Note that in the last line we used

$$\binom{k+1}{2} - k = \binom{k}{2}.$$

Fourth Variation. Determine the number of sequences $(x_1, x_2, \ldots, x_k)$ such that $x_i \in \{1, 2, 3\}$ and $x_1 + x_2 + \ldots + x_k = n$, where k and n are positive integers. In Step (2) and (3), we write $S_i = \{1, 2, 3\}$ and

$$\Phi_{S_i}(x) = x + x^2 + x^3.$$

Together with Steps (1) and (4), the product lemma, we get

$$\Phi_S(x) = (x + x^2 + x^3)^k.$$

By the Binomial Theorem in Step (5),

$$\Phi_S(x) = x^k \sum_{j=0}^{k} \binom{k}{j} (x + x^2)^j.$$

We still can't read off $[x^n]\Phi_S(x)$, so we use the binomial theorem again inside the sum (i.e. apply Step (5) again):

$$\Phi_S(x) = x^k \sum_{j=0}^{k} \sum_{i=0}^{j} \binom{k}{j} \binom{j}{i} x^{i+j}.$$

To get the coefficient of x^n, let $i + j = n - k$. This is the same as putting $i = n - k - j$, so in Step (6) we get

$$[x^n]\Phi_S(x) = \sum_{j=0}^{k} \binom{k}{j} \binom{j}{n-k-j}.$$

We don't need Step (7), so this is the answer.

2.8.2 Unrestricted number of parts

A natural question is the number of compositions of n into any number of parts. In other words, we want to count how many sequences of positive integers add up to n, regardless of the length of the sequence. We already see that this is just

$$\sum_{k=1}^{n}\binom{n-1}{k-1}=2^{n-1}$$

when $n \geq 1$. But we give an approach which generalizes using generating functions. Step (1) is to write

$$S=\bigcup_{k=0}^{\infty}\{(x_1,x_2,\ldots,x_k): x_i \geq 1, i \in [k]\}.$$

Notice the difference here, we have a union over k, which denotes the possible number of parts in the composition. Step (2) is to write $S_i = \mathbb{Z}^+$, the set of positive integers, and Step (3) is

$$\Phi_{S_i}(x)=x+x^2+x^3+\ldots=\frac{x}{1-x}.$$

Now in Step (4) we have to find $\Phi_S(x)$. Since S is defined by a union, we cannot just apply the product lemma; we have to apply the sum rule. So we get

$$\Phi_S(x)=\sum_{k=0}^{\infty}\prod_{i=1}^{k}\Phi_{S_i}(x)=\sum_{k=0}^{\infty}\left(\frac{x}{1-x}\right)^k=\frac{1}{1-\frac{x}{1-x}}=\frac{1-x}{1-2x}.$$

Here we used the geometric series formula. So the key point is that Step (4) changes slightly:

(4) *Use the sum lemma and product lemma to write* $\Phi_S(x)=\sum_{k=0}^{\infty}\prod_{i=1}^{k}\Phi_{S_i}(x)$.

The rest of the procedure is the same: in Step (5) we obtain

$$\Phi_S(x)=(1-x)(1-2x)^{-1}=(1-x)\sum_{j=0}^{\infty}(2x)^j=\sum_{j=0}^{\infty}2^jx^j-\sum_{j=0}^{\infty}2^jx^{j+1}.$$

In Step (6), we want $[x^n]\Phi_S(x)$. Well x^n appears when $j=n$ in the first sum and when $j=n-1$ in the second, in which case we get

$$[x^n]\Phi_S(x)=2^n-2^{n-1}=2^{n-1}.$$

Step (7) does not apply, and we are done.

Variation. We determine the number of ways an integer n can be written as a sum of 1s and 2s. In Step (1), we write

$$S=\bigcup_{k=0}^{\infty}\{(x_1,x_2,\ldots,x_k): x_i \in \{1,2\}, i \in [k]\}.$$

Then $S_i=\{1,2\}$ in Step (2) and in Step (3)

$$\Phi_{S_i}(x)=x+x^2.$$

In Step (4), using the sum and product lemmas, we get

$$\Phi_S(x) = \sum_{k=0}^{\infty}(x+x^2)^k = \frac{1}{1-x-x^2}$$

via geometric series. By the binomial theorem applied inside the sum below, Step (5) gives

$$\Phi_S(x) = \sum_{k=0}^{\infty} x^k(1+x)^k = \sum_{j=0}^{k}\binom{k}{j}x^{j+k}.$$

Letting $j = n-k$ in Step (6) for $k \leq n$, we get $[x^n]\Phi_S(x) = \sum_{k=0}^{n}\binom{k}{n-k}$.

2.8.3 Restricted subsets*

In the last few sections we counted sequences which add up to n. We will show how this allows us to count subsets of n with restrictions on the difference between consecutive elements of the subset when the elements are arranged in increasing order. For example, suppose we want to count sets of size k in $[n]$ such that no two elements are consecutive integers. For $k = 3$ and $n = 6$, the sets are

$$\{1,3,5\} \quad \{1,3,6\} \quad \{1,4,6\} \quad \{2,4,6\}.$$

There are four such sets. Now here is the idea: if we arrange the elements in increasing order (we did this already in the listing above), say $x_1 < x_2 < x_3$, then we can form a sequence (y_1, y_2, y_3, y_4) with $y_1 = x_1$, $y_2 = x_2 - x_1$, $y_3 = x_3 - x_2$, and $y_4 = 6 - x_3$. Observe that $y_1 \geq 1$, $y_2 \geq 2$, $y_3 \geq 2$ and $y_4 \geq 0$, and most importantly

$$y_1 + y_2 + y_3 + y_4 = x_1 + (x_2 - x_1) + (x_3 - x_2) + (6 - x_3) = 6.$$

So (y_1, y_2, y_3, y_4) is exactly a composition of 6 into four parts. This idea generalizes: if we want to count sets $\{x_1, x_2, \ldots, x_k\} \subseteq [n]$ such that $x_{i+1} - x_i \in S_{i+1}$ and $x_1 \in S_1$, then this is the same as counting compositions of n into $k+1$ parts such that the first part is a positive integer, the ith part is in S_i for $i \in [k]$, and the $(k+1)$th part is a non-negative integer. For example, if we apply our procedure to the problem above, we have $S_1 = \{1, 2, 3, \ldots\}$, $S_i = \{2, 3, 4, \ldots,\}$ for $2 \leq i \leq k$, and therefore if

$$S = \{(y_1, y_2, y_3, \ldots, y_{k+1}) : y_i \in S_i, i \in [k], y_{k+1} \geq 0\}$$

then the procedure for compositions gives

$$\Phi_S(x) = \frac{x}{1-x} \cdot \left(\frac{x^2}{1-x}\right)^{k-1} \cdot \frac{1}{1-x} = x^{2k-1}(1-x)^{-k-1}.$$

At Step (5) and (7) we see

$$\Phi_S(x) = x^{2k-1}\sum_{j=0}^{\infty}\binom{k+j}{k}x^j.$$

Finally in Step (6), with $j = n - 2k + 1$,

$$[x^n]\Phi_S(x) = \binom{n-k+1}{k}.$$

A quick check for $n = 6$ and $k = 3$ shows that this agrees with the listing above.

2.9 Binary strings

The number of binary strings of length n is 2^n, for each positive integer n. In this section, we count the number of binary strings with restrictions on the contiguous substrings which are allowed to occur. The main idea is to use the canonical block decomposition of a binary string. A block in binary string is a maximal nonempty substring of consecutive zeroes, or a maximal nonempty substring of consecutive ones. For example, the string below has six blocks, and they are each boxed.

$$\boxed{1111}\boxed{0}\boxed{11}\boxed{00000}\boxed{1}\boxed{0}.$$

Every string can be partitioned into its blocks in a unique way, and we refer to this as the block decomposition of a string. To represent this concept mathematically, we introduce some new notation. The concatenation of binary strings a and b is the binary string ab consisting of a followed by b. We denote by ε the empty string, so that $a\varepsilon = a$ for any binary string a. If A and B are sets of binary strings, then

$$AB := \{ab : a \in A, b \in B\}$$

is the concatenation of A and B. Define

$$A^* := \{\varepsilon\} \cup A \cup AA \cup \dots$$

which is the set of all concatenations of strings in A together with the empty string. We may call this the $*$-operation on a set. For example, $\{1\}^*$ consists of the empty string and blocks of ones of all possible lengths. It should be clear now that $\{0,1\}^*$ is the set of all binary strings. Notice also that $(\{0\}^*\{1\}^*)^*$ is also the set of all binary strings. As another example: $\{0\}^*\{1\}^*$ creates all strings which comprise a (possibly empty) block of zeroes follows by a (possibly empty) block of ones. So the strings in this set are $\varepsilon, 0, 1, 00, 11, 01, 000, 001, 011, 111$, and so on.

Given an expression involving concatenations and $*$-operations on sets of binary strings, we can ask whether a string in the resulting set of binary strings can be created in only one way. For example, the string 101 can be created in two ways from AB when $A = \{10, 1\}$ and $B = \{1, 01\}$ – we can concatenate $10 \in A$ with $1 \in B$ or $1 \in A$ with $01 \in B$. On the other hand, 101 is created in only one way from AB when $A = \{10, 1\}$ and $B = \{0, 01\}$. In the example $(\{0\}^*\{1\}^*)^*$, there are many ways to create a given string. For example, the string 011 can be created as the concatenation of 01 and 1 or the concatenation of 0 and 11. We say that the strings in a set involving concatenations and $*$-operations are uniquely created if the strings can be created in only one way. Clearly $\{0,1\}^*$ uniquely creates all binary strings, but $(\{0\}^*\{1\}^*)^*$ does not, as we observed above. The following theorem is essentially the fact that the block decomposition of a string is unique.

Theorem 9 *The strings in the identity below*

$$\{0,1\}^* = \{1\}^*(\{0\}\{1\}^*)^* = \{0\}^*(\{1\}\{0\}^*)^*$$

are uniquely created.

Proof ▷ We just consider $\{1\}^*(\{0\}\{1\}^*)^*$, the rest is similar. Now every string $a \in \{0,1\}^*$ can be represented uniquely as $a_1 0 a_2 0 a_3 \dots 0 a_k$ where a_i is a block of ones, or $a_i = \varepsilon$, for $i \in [k]$. Since $a_i \in \{1\}^*$, we see that $0a_i \in \{0\}\{1\}^*$ and so $a \in \{1\}^*(\{0\}\{1\}^*)^*$ is uniquely created. ∎

It is very important to see when sets of strings are uniquely created, for then we can count the number of strings in the set. For example, if the strings in AB are uniquely created, then $|AB| = |A||B|$. Using this concept, we are able to use the product and sum lemmas together with generating functions to count binary strings with restrictions on their substrings.

If S is a set of binary strings, and we defined $\omega(\sigma) = \ell(\sigma)$, the length of σ for $\sigma \in S$, then we denote as usual $\Phi_S(x)$ the generating function for this set of strings. Let's get a feeling for $\Phi_S(x)$ for various sets of strings. If $S = \{11\}$, then there are zero strings of length $0, 1, 3, 4, \dots$, so the coefficient of x^n in the generating function is zero if $n = 0, 1, 3, 4, \dots$. The set has exactly one string of length two, so the coefficient of x^2 is one, and $\Phi_S(x) = x^2$. Note that this is the same as the generating function for the sets $\{01\}$, $\{10\}$ and $\{00\}$, since all we care about is the number of strings of each length. Here is another example: suppose $S = \{\varepsilon, 0, 1, 10, 100, 1000, 10000, \dots\}$. In this case, there is exactly one string of length n for $n = 0, 2, 3, \dots$ and two strings of length $n = 1$. So

$$\Phi_S(x) = 1 + 2x + x^2 + x^3 + x^4 + \dots = \frac{1}{1-x} + x$$

where we used the familiar formula for geometric series. In general, we might be given a very complicated set of strings and then finding the generating function is tricky. But if the set S can be written in terms of concatenations and $*$-operations on smaller sets, and the strings are uniquely created, then it turns out we can determine the generating function for S in terms of the smaller sets. That will be our approach for the rest of this section: break down a complicated set of strings using concatenations and $*$-operations, and then find their generating functions to get the generating function for S. The precise details are given by the next theorem:

Theorem 10 *Let A and B be sets of binary strings. If the elements of AB are uniquely created, then*

$$\Phi_{AB}(x) = \Phi_A(x) \cdot \Phi_B(x).$$

If the elements of A^* are uniquely created, then

$$\Phi_{A^*}(x) = \frac{1}{1 - \Phi_A(x)}.$$

Proof ▷ Let $A^k = AAA \cdots A$ (k times), where $A^0 = \{\varepsilon\}$. Since $A^* = \{\varepsilon\} \cup A \cup AA \cup \dots$ we have from the first part

$$\Phi_{A^*}(x) = \sum_{k=0}^{\infty} \Phi_{A^k}(x) = \sum_{k=0}^{\infty} \Phi_A(x)^k.$$

This is a geometric series, which gives the second identity. For the first identity,

$$\Phi_{AB}(x) = \sum_{ab \in AB} x^{\omega(ab)} = \sum_{a \in A} \sum_{b \in B} x^{\omega(a)} x^{\omega(b)} = \Phi_A(x)\Phi_B(x)$$

since $\omega(ab) = \omega(a) + \omega(b)$. ∎

Let's work out the generating function for some sets of strings using the theorem. We know that the strings in $S = \{00\}^*\{11\}^*$ are uniquely created. Therefore Theorem 10 applies to show

$$\Phi_S(x) = \frac{1}{1 - \Phi_{\{11\}}(x)} \cdot \frac{1}{1 - \Phi_{\{00\}}(x)}.$$

Since $\Phi_{\{11\}}(x) = x^2 = \Phi_{\{00\}}(x)$, the generating function is

$$\Phi_S(x) = \frac{1}{(1 - x^2)^2}.$$

Here we caution that the generating function for $\{0, 1\}$ and the generating function for $\{01\}$ are $2x$ and x^2 respectively: remember always that the coefficient of x^n is the number of strings of length n in the given set.

We use Theorem 10 with Theorem 9 to count binary strings with restrictions on the substrings. The simplest case is to count the strings of length n in $\{0, 1\}^*$. Since the strings in $S = \{0, 1\}^*$ are uniquely created, we have

$$\Phi_S(x) = \frac{1}{1 - \Phi_{\{0,1\}}(x)} = \frac{1}{1 - 2x}$$

by the second identity in Theorem 10. Since this gives

$$\Phi_S(x) = \sum_{j=0}^{\infty} 2^j x^j$$

the number of strings of length n in S is 2^n, as expected. Throughout the next section, we will be content in finding the generating function for sets of strings, without actually determining the coefficients in the generating function.

2.9.1 Restrictions on substrings

Consider counting binary strings of length n with no two consecutive ones. By Theorem 9, all strings are uniquely created in the set $\{1\}^*(\{0\}\{1\}^*)^*$. We can extract from this an expression for the strings we want: since the $\{1\}^*$ terms represent the blocks of ones, we can replace the occurrences of $\{1\}^*$ by $\{\varepsilon, 1\}$ to express that every block of ones has length at most one. So if S is the set of such strings, then

$$S = \{\varepsilon, 1\}(\{0\}\{\varepsilon, 1\})^*$$

and the strings in S are uniquely created. Therefore by Theorem 9

$$\Phi_S(x) = \frac{1 + x}{1 - x - x^2}.$$

We could try to determine $[x^n]\Phi_S(x)$, but we will develop methods in the next section to do this more easily than applying the binomial theorem.

Variation 1. Here is another example: suppose we only allow blocks of ones of even length. Then in Theorem 9 we can write

$$S = \{\varepsilon, 11, 1111, \ldots\}(\{0\}\{\varepsilon, 11, 1111, \ldots\}^*)^*$$

for the set of such strings. By Theorem 10,

$$\Phi_S(x) = \frac{1 + x^2 + x^4 + \ldots}{1 - x(1 + x^2 + x^4 + \ldots)} = \frac{\frac{1}{1-x^2}}{1 - \frac{x}{1-x^2}} = \frac{1}{1 - x - x^2}.$$

This is the generating function, from which we can compute $[x^n]\Phi_S(x)$, the number of strings of length n where every block of ones has even length.

Theorem 9 is somewhat restrictive, since it only allows placing a restriction on blocks of ones or restriction on blocks of zeroes, but not both. For example, we might disallow two zeroes followed by two ones. The answer is to come up with a finer representation of $\{0,1\}^*$ in terms of double block decompositions. Observe that every binary string can be decomposed into double blocks, where a double block consists of a block of zeroes followed by a block of ones. Such a decomposition is shown below:

a | 000001 | 001 | 01 | 01 | 01 | 00001 | 01 | 001111111 | 0001111 | b

where a and b are (possibly empty) sequences of ones and zeroes respectively. This observation is the proof of the following theorem:

Theorem 11 *The strings in the identities below are uniquely created*

$$\{0,1\}^* = \{0\}^*(\{1\}\{1\}^*\{0\}\{0\}^*)^*\{1\}^* = \{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*.$$

The middle brackets in the expressions in the theorem represents the generation of the double blocks in a binary string: a double block by definition is an element of $\{0\}\{0\}^*\{1\}\{1\}^*$. We now apply this result to the set S of binary strings with no two zeroes followed by two ones. In other words, 0011 is not allowed as a substring. In the brackets in Theorem 11, the double blocks of S are the strings in the set

$$\{0\}\{\varepsilon\}\{1\}\{1\}^* \cup \{0\}\{0, 00, 000, \ldots\}\{1\}\{\varepsilon\}.$$

In words, a double block either has the form $01111\ldots$ or has the form $00000\ldots01$. Therefore

$$S = \{1\}^*(\{0\}\{\varepsilon\}\{1\}\{1\}^* \cup \{0\}\{0, 00, 000, \ldots\}\{1\}\{\varepsilon\})^*\{0\}^*.$$

The generating function for S is, by Theorem 10

$$\Phi_S(x) = \Big(\frac{1}{1-x}\Big)^2 \cdot \frac{1}{1 - \frac{x^2}{1-x} - \frac{x^3}{1-x}}.$$

While we could in theory find $[x^n]\Phi_S(x)$, we will not do it here.

Variation 2. We consider a second example: suppose S is the set of binary strings where every block has even length. Since $\{00\}^*$ denotes a block of zeroes of even length, and similarly for $\{11\}^*$, we obtain in the manner of Theorem 11

$$S = \{11\}^*(\{00\}\{00\}^*\{11\}\{11\}^*)^*\{00\}^*.$$

Note that all the double blocks have at least two zeroes and two ones, except perhaps for the first and last which are allowed to be empty. Now by Theorem 10

$$\Phi_S(x) = \left(\frac{1}{1-x^2}\right)^2 \cdot \frac{1}{1-\left(\frac{x^2}{1-x^2}\right)^2} = \frac{1}{(1-x^2)^2 - x^4} = \frac{1}{1-2x^2}.$$

This time let's find the answer. Clearly

$$\Phi_S(x) = \sum_{j=0}^{\infty} 2^j x^{2j}$$

and setting $j = n/2$ if n is even, we get $[x^n]\Phi_S(x) = 2^{n/2}$. So our answer is that $2^{n/2}$ is the number of strings of length n where every block has even length, provided n is even, otherwise the answer is zero. This is to be expected: if we delete every second entry of such a string, we get a binary string of length $n/2$, and this defines a bijection from the strings of length n where every block has even length to the binary strings of length $n/2$, of which there are $2^{n/2}$.

Variation 3. Here is a final and slightly more tricky example. We will determine the generating function for the set S of strings which do not contain 101. Consider first the set T of strings in S which have no double block (recall a double block is a non-empty string of zeroes followed by a non-empty string of ones). These strings are represented in Theorem 11 by $\{1\}^*\{\varepsilon\}\{0\}^*$, since none of them could ever contain 101. Now consider those strings in S which have at least one double block. By Theorem 11, the set of all strings which have at least one double block is created uniquely by

$$\{1\}^*(\{0\}\{0\}^*\{1\}\{1\}^*)(\{0\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*.$$

We have to extract from this the strings that are in S. The strings in S starting with a one are given by

$$S_1 \quad = \quad \{1, 11, 111, \ldots\}(\{00\}\{0\}^*\{1\}\{1\}^*)(\{00\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*.$$

This is because every double block in S_1 begins with at least two zeroes. Next let S_0 be the strings in S starting with zero. Then

$$S_0 \quad = \quad \{\varepsilon\}(\{0\}\{0\}^*\{1\}\{1\}^*)(\{00\}\{0\}^*\{1\}\{1\}^*)^*\{0\}^*$$

since the first two blocks are arbitrary, but the ones after that have exactly one 0. Finally

$$S = T \cup S_0 \cup S_1$$

and the strings on the right are uniquely created. From this and Theorem 10, we can find the generating function. It is convenient to put $y = (\frac{x}{1-x})^2$. By the sum lemma,

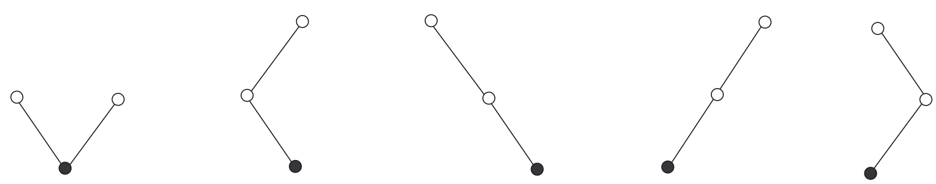
$$\begin{aligned}
\Phi_S(x) &= \Phi_T(x) + \Phi_{S_1}(x) + \Phi_{S_0}(x) \\
&= \frac{1}{(1-x)^2} + \frac{x^4}{(1-x)^4} \cdot \frac{1}{1-\frac{x^3}{(1-x)^2}} + \frac{x^2}{(1-x)^3} \cdot \frac{1}{1-\frac{x^3}{(1-x)^2}} \\
&= \frac{1}{(1-x)^2} + \frac{x^2(x^2+1)}{(1-x)^3 - x^2}.
\end{aligned}$$

2.10 Binary Trees*

A rooted binary tree is a tree with no nodes or a tree in which there is a fixed root node of degree two, and all other nodes have degree one or degree three. At every node, there is a left and a right branch. We wish to count the number of rooted binary trees with n vertices; we will prove the following theorem:

Theorem 12 *The number of rooted binary trees on n vertices is $\frac{1}{n+1}\binom{2n}{n}$ for $n \geq 0$.*

For example, all the rooted binary trees with three vertices are shown below:



Let ε denote the empty tree (the tree with no vertices). If S is the set of all rooted binary trees, then any non-empty tree in S can be split into two rooted trees by removing the original root of the tree. From all pairs of rooted binary trees, we can recreate all non-empty binary trees by adding a new root and joining it to the roots of each binary tree in the pair. Let's call this the join operation; if $s_1, s_2 \in S$ are two rooted trees, let $s_1 \vee s_2$ be their join. Let $S_1 \vee S_2$ denote the set of trees formed by joining each tree in S_1 with each tree in S_2. Note that by definition, $\varepsilon \vee \varepsilon$ is a single vertex. We have argued that $S \vee S = S \backslash \{\varepsilon\}$. Define the weight of a tree $s \in S$ to be $\omega(s) = |s|$, the number of vertices in s, and let $\Phi_S(x)$ denote the generating function for S with respect to this weight function. The following lemma is very similar to the product lemma for cartesian products of sets, but the key difference between the cartesian product and the join is that the join of two trees is a tree with one more vertex (the new root). This adds an extra x in the formula for the generating function for the join of two trees:

Lemma 13 *Let S_1, S_2 be two sets of binary trees. Then*

$$\Phi_{S_1 \vee S_2}(x) = x\Phi_{S_1}(x)\Phi_{S_2}(x).$$

Proof ▷ By definition of generating functions,

$$\begin{aligned}
\Phi_{S_1 \vee S_2}(x) &= \sum_{s \in S_1 \vee S_2} x^{\omega(s)} \\
&= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} x^{\omega(s_1 \vee s_2)} \\
&= \sum_{s_1 \in S_1} \sum_{s_2 \in S_2} x^{\omega(s_1)+\omega(s_2)+1}
\end{aligned}$$

The key point here is that $\omega(s_1 \vee s_2) = \omega(s_1) + \omega(s_2) + 1$, since we added a vertex when we join trees s_1 and s_2. Continuing with the calculation,

$$\begin{aligned}\Phi_{S_1 \vee S_2}(x) &= x \sum_{s_1 \in S_1} x^{\omega(s_1)} \sum_{s_2 \in S_2} x^{\omega(s_2)} \\ &= x\Phi_{S_1}(x)\Phi_{S_2}(x).\end{aligned}$$

This completes the proof. ∎

By the discussion above, $S = S \vee S \cup \{\varepsilon\}$ so by the lemma above, and the sum lemma,

$$\Phi_S(x) = \Phi_{\{\varepsilon\}}(x) + x\Phi_S(x)^2 = 1 + x\Phi_S(x)^2.$$

This can be rearranged to give

$$2x\Phi_S(x) - 1 = \pm(1-4x)^{\frac{1}{2}}.$$

We are after the coefficient of x^n in $\Phi_S(x)$. To find this, expand the right hand side as a sum: recall that

$$(1+x)^{\frac{1}{2}} = 1 + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{2^{2k-1}k} \binom{2k-2}{k-1} x^k$$

which means that after a calculation (check this as an exercise),

$$(1-4x)^{\frac{1}{2}} = 1 - 2x \sum_{k=1}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k.$$

Choosing the minus sign in the equation for $2x\Phi_S(x) - 1$, we get

$$2x\Phi_S(x) - 1 = 2x \sum_{k=1}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k - 1$$

and therefore

$$\Phi_S(x) = \sum_{k=1}^{\infty} \frac{1}{k+1} \binom{2k}{k} x^k.$$

The coefficient of x^n is $\frac{1}{n+1}\binom{2n}{n}$, and is the answer for the number of binary trees with n vertices. These numbers are called the Catalan numbers, and will come up in many more combinatorial problems. This proves Theorem 12.

2.11 Catalan Numbers*

The Catalan numbers, as we mentioned appear in many combinatorial problems. These include the ballot problem and the number of bracketings of n letters, as well as the number of paths of length $2n$ in the plane which stay above the x-axis and the circular non-crossing handshake problem. A proper description of these problems will be given at the end of this section. First we discuss Euler's Polygon Division problem, which asks for the number of ways to divide a convex n-gon into triangles with diagonals i.e. the number of triangulations of a convex polygon with n sides. This is done for $n = 4, 5, 6$ in the figure below:

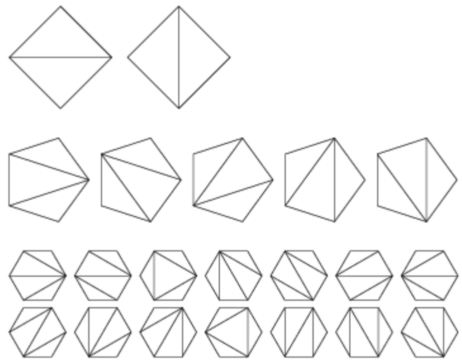


Figure : Triangulations of a polygon

We will show that the solution to Euler's problem is the Catalan number $\frac{1}{n-2}\binom{2n-4}{n-2}$ for $n \geq 3$ by reducing the problem to counting rooted binary trees. Draw the polygon as a regular polygon, P_n, with the lowest side parallel to the x-axis. Now create a rooted binary tree as follows: inside each triangle of the triangulation, draw a vertex, and join two vertices with an edge if their corresponding triangles share a side. Let's prove that these vertices and edges form a tree. We prove this by strong induction on $n \geq 3$. For $n = 3$, P_3 is a triangle and only one vertex is drawn, so we obtain a tree. Suppose $n > 3$. Then at least one diagonal is drawn in P_n. Partition P_n into two triangulated polygons P_{n_1} and P_{n_2} by cutting the original polygon along a diagonal. By induction, the vertices and edges drawn inside P_{n_1} and inside P_{n_2} form trees, T_1 and T_2, respectively. Now only one more edge is drawn inside P_n, joining T_1 and T_2. It is clear that joining two trees with an edge gives a new tree, so this completes the proof. So inside P_n we have drawn a tree, T, and the tree has $n-2$ vertices. Furthermore, every vertex drawn in P_n has degree at most three, since it is only joined to vertices in triangles which share a side with its own triangle. We conclude that T is a binary tree with $n-2$ vertices, and we can root the tree at the vertex inside the triangle one of whose sides is the lowest side of P_n – see the illustration.

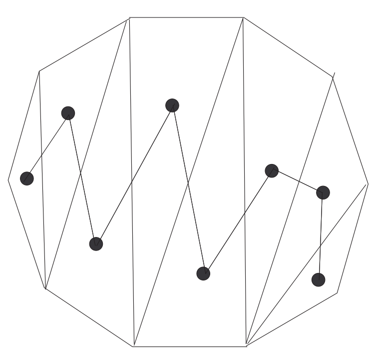


Figure : Trees from triangulations

The number of rooted binary trees on $n-2$ vertices is $\frac{1}{n-1}\binom{2n-4}{n-2}$ by Theorem 12, so this solves Euler's Division Problem.

2.12 Related Combinatorial Problems*

The other problems we have mentioned can be reduced in similar ways to counting binary trees. There are many other problems involving Catalan Numbers which we do not mention here.

Ballot Problem. The ballot problem is the number of possible processes of $2n$ votes for candidates A and B such that at any stage in the voting process, candidate A has at least as many votes as candidate B.

Path Problem. This problems consists in determine the number of paths $(x_1, x_2, \ldots, x_{2n})$ in the plane such that $x_1 = (0, 0)$, and for any $i < 2n$, $x_{i+1} = x_i + (1, -1)$ or $x_{i+1} = x_i + (1, 1)$, and the path never dips below the x-axis. In other words, at any step, we move one step up and one step right, or one step down and one step right, and we never go below $y = 0$.

Binary Bracketing Problem. Suppose we are given a list $x_1x_2x_3 \ldots x_{2n}$ of letters. In how many ways can be place $2n$ brackets (or) between these letters in a sensible mathematical way? In particular, the number of right brackets (reading from left to right) should never exceed the number of left brackets, and the total number of left and right brackets is equal. For example,

$$\begin{array}{ccc} (x_1(x_2x_3))x_4 & x_1(x_2(x_3))x_4 & x_1(x_2(x_3))x_4 \\ & x_1(x_2(x_3x_4)) & x_1(x_2(x_3)x_4) \end{array}$$

are the five valid bracketings when $n = 2$.

Handshake Problem. If $2n$ people are sitting around a table, and each person shakes hands with exactly one other person, how many ways can people shake hands if no people have to cross arms to shake hands? A possible configuration of the handshakes is shown below:

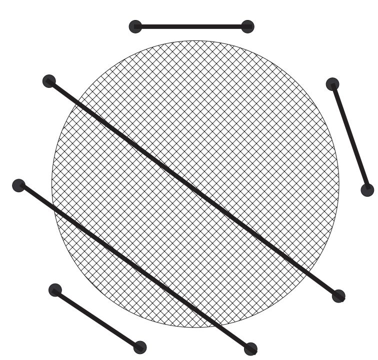


Figure : Non-crossing handshakes

2.13 Exercises

Question 1. Determine the exact numerical values of the following binomial coefficients:

(a) $\binom{1}{2}$
(b) $\binom{2}{0}$
(c) $\binom{n}{n-1}$
(d) $\binom{1/2}{4}$
(e) $\binom{-1/2}{4}$
(f) $\binom{-1/3}{3}$
(g) $\binom{1/3}{k}$.

Question 2. Find the inverses of the following formal power series as a sum of powers of x, or state that the power series has no inverse. Justify your answers.

(a) $1 - 2x + x^2$
(b) $x(1 - x)$
(c) $4 - x^2$
(d) $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$
(e) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$
(f) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$.

Question 3. Determine the generating function in closed form for the given set S of configurations with weight function ω.

(a) S is the set of subsets of $[n]$, $\omega(\sigma) = |\sigma|$
(b) $S = \mathbb{Z}^+$ and $\omega(\sigma) = 2\sigma$.
(c) $S = \mathbb{Z}^+$ and $\omega(\sigma) = \sigma$ if σ is odd and $\omega(\sigma) = 0$ otherwise.
(d) $S = [n]$ and $\omega(\sigma) = 1$ if σ is odd and $\omega(\sigma) = 0$ if σ is even.
(e) S is the set of permutations of $[4]$, $\omega(\sigma)$ is the number of fixed points of σ.
(f) S is the set of pairs (a, b) of positive integers, and $\omega(a, b) = a + b$.
(g) S is the set of pairs (a, b) of positive integers, and $\omega(a, b) = 2a + b$.

Question 4. Compute the average maximum element of non-empty subsets of $[n]$.

Question 5. What is the average difference between the largest and smallest elements of a non-empty subset of $[n]$?

Question 6. Determine the number of compositions of n into k parts with the given restrictions.

(a) Each part is a positive even integer.
(b) Each part is an element of $\{2, 3\}$
(c) The ith part is at most i
(d) Exactly one part is odd, the rest are positive and even.

Question 7. Determine the number of compositions of n into any number of parts, where each part is odd.

Question 8. How many sets $\{x_1, x_2, \ldots, x_k\} \subset [n]$ have $x_{i-1} + i \leq x_i$ for $2 \leq i \leq k$?

Question 9* Let S denote the set of subsets of positive integers of size k, with weight function $\omega(\{x_1, x_2, \ldots, x_k\}) = x_1 + x_2 + \ldots + x_k$. Prove that

$$\Phi_S(x) = \prod_{r=1}^{\infty} \frac{1}{1 - x^r}.$$

Question 10. Determine the number of binary strings of length n with the given restrictions.

(a) The strings have only blocks of odd length.
(b) The strings do not contain 011.
(c) The strings do not contain 101 or 010.
(d) The strings do not contain 101.

Question 11. Determine the generating function for the set of binary strings of all lengths not containing 011.

Question 12. Determine a formula for the generating function of the set of k-ary trees: these are trees with a root vertex and in which every vertex has at most k children.