

### 3 Recurrence Equations

The generating function for the set of binary strings with no block of ones of odd length was shown to be

$$\Phi(x) = \frac{1}{1 - x - x^2}$$

in the last section. We could find the coefficient of  $x^n$  in  $\Phi(x)$  using the binomial theorem, and the answer would be

$$[x^n]\Phi(x) = \sum_{j=0}^n \binom{j}{n-j}.$$

It would be nicer if the answer were written in closed form (i.e. without a sum). It turns out that this is possible using recurrence relations. Let's outline the approach for the example under discussion, and then generalize. We know that we are seeking  $a_n = [x^n]\Phi(x)$  in

$$\Phi(x) = \sum_{k=0}^{\infty} a_k x^k = \frac{1}{1 - x - x^2}.$$

Multiply both sides by  $1 - x - x^2$  to get

$$\sum_{k=0}^{\infty} a_k x^k - \sum_{k=0}^{\infty} a_k x^{k+1} - \sum_{k=0}^{\infty} a_k x^{k+2} = 1.$$

Now  $x^n$  appears in the first sum when  $k = n$ , in the second sum when  $k = n - 1$  and in the third sum when  $k = n - 2$ , provided  $n \geq 2$  (since  $k$  can't be negative). This means that the coefficient of  $x^n$  on the left hand side is  $a_n - a_{n-1} - a_{n-2}$  whenever  $n \geq 2$ . But the coefficient of  $x^n$  on the right hand side is zero, so we get the equation

$$a_n - a_{n-1} - a_{n-2} = 0.$$

This is our first example of a recurrence relation; it is actually quite a famous one, called the Fibonacci equation. Since  $a_0 = a_1 = 1$  (the empty string has no block of ones of odd length, and the string 0 is the only string of length one with no block of ones of odd length) we can work out

$$\begin{aligned} a_2 &= a_1 + a_0 = 2 \\ a_3 &= a_2 + a_1 = 3 \\ a_4 &= a_3 + a_2 = 5 \\ a_5 &= a_4 + a_3 = 8 \\ a_6 &= a_5 + a_4 = 13 \\ a_7 &= a_6 + a_5 = 21 \\ a_8 &= a_7 + a_6 = 34 \end{aligned}$$

and in principle we could find  $a_n$  for any given value of  $n$ . This is not much better than the sum we had for  $a_n$ , but it is a step in the right direction. We promised a closed form function of  $n$  for  $a_n$ , and to find this we have to solve the equation  $a_n = a_{n-1} + a_{n-2}$  with  $a_0 = a_1 = 1$ . Here is the key idea: we guess that  $a_n = x^n$  for some value of  $x$  and then try to find the value of  $x$ . To find the value of  $x$ , substitute  $x^n$  into the equation and divide by  $x^{n-2}$  to get

$$x^2 - x - 1 = 0.$$

Solving for  $x$ , we get via the quadratic formula

$$x = \frac{1 + \sqrt{5}}{2} \quad \text{or} \quad x = \frac{1 - \sqrt{5}}{2}.$$

But then it is easy to see that

$$a_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

is also a solution to the equation for any real values  $c_1, c_2$  (check this by direct substitution). However since  $a_0 = a_1 = 1$ , the values of  $c_1$  and  $c_2$  are not arbitrary: we must have

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_1 \left( \frac{1 + \sqrt{5}}{2} \right) + c_2 \left( \frac{1 - \sqrt{5}}{2} \right) &= 1. \end{aligned}$$

These equations give

$$c_1 = \frac{1 + \sqrt{5}}{2\sqrt{5}} \quad \text{and} \quad c_2 = \frac{\sqrt{5} - 1}{2\sqrt{5}}.$$

So the final answer is

$$a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}.$$

This is really a remarkably simple answer for  $a_n$ , and certainly a lot more computable than the answer we had before, namely

$$a_n = \sum_{j=0}^n \binom{j}{n-j}.$$

We have concluded that an approach to finding the coefficient of  $x^n$  in the generating function  $\Phi(x)$  is to solve the equation  $a_n = a_{n-1} + a_{n-2}$  where  $a_0 = a_1 = 1$ . Such an equation is called a homogeneous linear recurrence equation, and we are now in a position to solve even more general homogeneous equations.

A homogeneous linear recurrence relation of order  $k$  is an equation of the form

$$b_k a_n + b_{k-1} a_{n-1} + \dots + b_0 a_{n-k} = 0$$

where  $b_0, b_1, \dots, b_k$  are real numbers and we have to solve for  $a_n$ . For example, the equation  $a_n - a_{n-1} - a_{n-2} = 0$  in the last section is an equation of order two, with  $b_2 = 1, b_1 = -1$  and  $b_0 = -1$ . This is a particularly famous equation, called the Fibonacci equation, and we solved it in the last section. The method we used to solve the Fibonacci equation generalizes to homogeneous linear equations. The idea is to try the solution  $a_n = x^n$ , in which case we get

$$b_k x^k + b_{k-1} x^{k-1} + \dots + b_0 = 0.$$

This important equation is called the characteristic equation, and determines the behaviour of  $a_n$ , via the theorem below. First we need some terminology. The degree of a polynomial  $P(x)$  is the exponent of the largest power of  $x$  appearing in the polynomial. Every polynomial  $P(x)$  of degree  $k$  can be written as

$$(x - \alpha_1)^{m_1} (x - \alpha_2)^{m_2} \dots (x - \alpha_j)^{m_j}$$

where  $\alpha_1, \alpha_2, \dots, \alpha_j$  are all distinct complex numbers and  $m_1 + m_2 + \dots + m_j = k$ . We define  $m_i$  to be the multiplicity of the root  $\alpha_i$  of the equation. For example,  $(x-1)^2(x-2)(x-3)$  is a polynomial of degree four with  $m_1 = 2$ ,  $m_2 = 1$  and  $m_3 = 1$ .

**Theorem 1** *Let  $b_k a_n + b_{k-1} a_{n-1} + \dots + b_0 a_{n-k} = 0$  be a homogeneous recurrence equation. Suppose that  $\alpha_1, \alpha_2, \dots, \alpha_j$  are the distinct roots of the characteristic equation, where  $\alpha_i$  has multiplicity  $m_i$ . Then the solution to the recurrence equation is*

$$a_n = c_1(n)\alpha_1^n + c_2(n)\alpha_2^n + \dots + c_j(n)\alpha_j^n$$

where  $c_1, c_2, \dots, c_j$  are polynomials in  $n$  of degree at most  $m_i - 1$ .

The polynomials  $c_1, c_2, \dots, c_j$  can be determined if we know the value of  $a_0, a_1, \dots, a_{k-1}$ : in general we need  $k$  specifications on  $a_n$  when the characteristic equation has degree  $k$ , in order to specify uniquely the values of  $c_1, c_2, \dots, c_j$ . These specifications are called the initial conditions of the recurrence equation, for example, the initial conditions we had for  $a_n - a_{n-1} - a_{n-2} = 0$  were  $a_0 = a_1 = 1$  and the characteristic equation was  $x^2 - x - 1 = 0$ . Theorem 1 can be proved using generating functions and partial fractions, but we will not do it here. The theorem gives a general method for solving recurrence equations, which we summarise by the following steps:

### Method for Solving Recurrence Equations

- (1) Write down the characteristic equation
- (2) Solve the characteristic equation
- (3) Write down the general form of the solution  $a_n$  given in Theorem 1
- (4) Determine the polynomials  $c_1, c_2, \dots, c_j$  in Theorem 1 using the initial conditions on  $a_0, a_1, \dots$

Let's review this procedure for the equation  $a_n = a_{n-1} + a_{n-2}$ . The characteristic equation is  $x^2 - x - 1 = 0$ , which as we saw has roots  $\alpha_1 = \frac{1+\sqrt{5}}{2}$  and  $\alpha_2 = \frac{1-\sqrt{5}}{2}$ . This gives the general form for  $a_n$  that we had before:

$$a_n = c_1 \alpha_1^n + c_2 \alpha_2^n.$$

According to Theorem 1,  $c_i$  is a polynomial of degree at most  $m_i - 1$ , but  $m_i = 1$  in this case, so  $c_i$  is just a constant. Then, as before, we work out the value of  $c_1$  and  $c_2$  using  $a_0 = a_1 = 1$ .

**Example 1.** Suppose we want to solve  $a_n = a_{n-1} + 2a_{n-2}$  where  $a_0 = 1$  and  $a_1 = 2$ . The characteristic equation is  $x^2 - x - 2 = 0$  so  $x = -1$  or  $x = 2$ . Therefore

$$a_n = c_1(-1)^n + c_2 2^n$$

where  $c_1, c_2$  are constants (since  $m_1 = m_2 = 1$ ). Now

$$1 = c_1 + c_2 \quad \text{and} \quad 2 = -c_1 + 2c_2$$

from the initial conditions. Solving these equations we get  $c_1 = 0$  and  $c_2 = 1$ , and so finally  $a_n = 2^n$ .

**Example 2.** Suppose we want to solve the equation  $a_n = 4a_{n-1} - 4a_{n-2}$ , where  $a_0 = 1$  and  $a_1 = 3$ . The characteristic equation is

$$x^2 - 4x + 4 = 0$$

and this has one root, namely  $x = 2$ , of multiplicity two. Therefore by Theorem 1 we know that

$$a_n = c_1(n)2^n$$

where  $c_1(n)$  is a polynomial of degree  $m_1 - 1 = 2 - 1 = 1$ . Now this means  $c_1(n) = cn + d$  where  $c$  and  $d$  are constants. So  $a_n = (cn + d)2^n$ , and  $c$  and  $d$  are determined by the initial conditions:

$$(c0 + d)2^0 = 1 \quad \text{and} \quad (c1 + d)2^1 = 3.$$

Solving for  $c$  and  $d$ , we get  $c = \frac{1}{2}$  and  $d = 1$ , so the solution is  $a_n = (\frac{1}{2}n + 1)2^n$ .

**Example 3.** Suppose we solve  $a_n = 3a_{n-1} - 3a_{n-2} + a_{n-3}$  where  $a_0 = 1$ ,  $a_1 = 3$ , and  $a_2 = 7$ . Then the characteristic equation is

$$x^3 - 3x^2 + 3x - 1 = 0$$

which is the same as  $(x - 1)^3 = 0$ . Thus  $x = 1$  is a root of multiplicity three, and

$$a_n = (an^2 + bn + c)1^n = an^2 + bn + c.$$

If we solve for  $a, b, c$  using the initial conditions, we get  $a = b = c = 1$ , so the final answer is

$$a_n = n^2 + n + 1.$$

In this and preceding examples, we can check the answer by substituting it into the original equation.

### 3.1 Asymptotics

In general, it is not possible to explicitly solve the characteristic equation. For example, if we consider the recurrence equation  $a_n = a_{n-1} + 2a_{n-3} - a_{n-4} + a_{n-5}$  we get the characteristic equation

$$x^5 - x^4 - 2x^2 + x - 1 = 0$$

and we cannot solve this explicitly. It turns out that  $\alpha_1 = 1.6483$  is the largest root of the equation in modulus, and all the other roots are much less than one in modulus. So if the roots are  $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$  then Theorem 1 tells us

$$a_n = c_1\alpha_1^n + c_2\alpha_2^n + c_3\alpha_3^n + c_4\alpha_4^n + c_5\alpha_5^n.$$

But  $\alpha_2, \alpha_3, \alpha_4, \alpha_5$  are less than one, and the  $c_i$ s are only polynomials in  $n$ , so if  $n$  is large enough,

$$a_n = [c_1\alpha_1^n].$$

Another way of saying this is that for  $i > 1$ ,

$$\lim_{n \rightarrow \infty} c_i\alpha_i^n = 0$$

so eventually these terms disappear into rounding  $c_1\alpha_1^n$  to the nearest integer. So the moral of the story is that if (by computer or otherwise) one of the roots of the characteristic equation is larger in modulus than all the others, then we take only the term in Theorem 1 corresponding to this root for  $a_n$ . In a similar way, when we had the equation  $a_n = a_{n-1} + a_{n-2}$ , we see that the smaller root of the characteristic equation is much less than a half, and deduce the nice formula

$$a_n = \left\lceil \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} \right\rceil.$$

For example, if we want to know  $a_8$ , the number of binary strings of length eight in which all blocks of ones have even length, then we compute

$$\frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} = 33.99411648\dots$$

and when we round this to the nearest integer we get  $a_8 = 34$ . This agrees with our directly computed answer for  $a_8$  given above.

### 3.2 Recurrences and generating functions

So far, our motivation for introducing recurrence equations was to be able to find  $[x^n]\Phi_S(x)$ , where  $\Phi_S(x)$  is a generating function for a combinatorial problem. While recurrences are interesting in their own right, we will now spend some time using them to solve combinatorial problems with generating functions. We consider the following example: determine the number of compositions of  $n$  with any number of parts such that each part is two or three. The generating function is checked to be

$$\sum_{n=0}^{\infty} a_n x^n = \Phi(x) = \sum_{k=0}^{\infty} (x^2 + x^3)^k = \frac{1}{1 - x^2 - x^3}.$$

Multiplying by  $1 - x^2 - x^3$ , we get

$$\sum_{n=0}^{\infty} a_n x^n - \sum_{n=0}^{\infty} a_n x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+3} = 1.$$

It follows by looking at the coefficient of  $x^n$  for  $n \geq 3$  that

$$a_n - a_{n-2} - a_{n-3} = 0.$$

The initial conditions come from the statement of the problem: clearly  $a_0 = 1$ ,  $a_1 = 0$ ,  $a_2 = 1$ . So we have found a recurrence equation, and initial conditions, which is all we need to apply Theorem 1. In this case, the characteristic equation is  $x^3 - x - 1 = 0$ , which does not have particularly nice roots. On the other hand, they are easy for a computer to handle, and in the end we just write down  $a_n$  using Theorem 1. This example illustrates the basic approach: first find the generating function in the form

$$\Phi(x) = \frac{P(x)}{Q(x)}$$

where  $P(x)$  and  $Q(x)$  are polynomials. If  $Q(x) = b_0x^k + b_1x^{k-1} + \dots + b_k$ , then the recurrence equation we get for  $a_n = [x^n]\Phi(x)$  is

$$b_k a_n + b_{k-1} a_{n-1} + \dots + b_0 = 0$$

and the initial conditions  $a_0, a_1, \dots, a_{k-1}$  are determined from the statement of the problem. Then Theorem 1 takes over and we solve for  $a_n$ . Notice that the characteristic equation is just given by the expression  $x^k Q(1/x)$ .

**Variation 1.** We count the number of binary strings of length  $n$  where every block has even length. In a preceding example (see Variation 2 Section 2.9.1), we saw that the generating function is

$$\Phi(x) = \frac{1}{1 - 2x^2}.$$

The corresponding recurrence equation is  $a_n - 2a_{n-2} = 0$ , with characteristic equation  $x^2 - 2 = 0$ . So the roots are  $\alpha_1 = \sqrt{2}$  and  $\alpha_2 = -\sqrt{2}$ , each of multiplicity one, and

$$a_n = c_1(\sqrt{2})^n + c_2(-\sqrt{2})^n$$

where  $c_1$  and  $c_2$  are constants. The initial conditions are  $a_0 = 1$  (the empty string has all its blocks of even length) and  $a_1 = 0$ . Therefore

$$c_1 + c_2 = 1 \quad \text{and} \quad \sqrt{2}c_1 - \sqrt{2}c_2 = 0.$$

It follows that  $c_1 = c_2 = \frac{1}{2}$  and therefore

$$a_n = \frac{1}{2}\sqrt{2}^n + \frac{1}{2}(-\sqrt{2})^n = 2^{n/2-1}(1 + (-1)^n).$$

If  $n$  is odd, this is zero, and if  $n$  is even, we get  $2^{n/2}$ , as expected.

**Variation 2.** We determine the number of binary strings of length  $n$  in which every block of zeroes has length at least two, or the string has no zeroes. Let  $S$  be the set of all such strings (of any length). Since

$$\{0, 1\}^* = \{0\}^* (\{1\}\{0\}^*)^*$$

and the strings here are uniquely created,

$$S = \{\varepsilon, 00, 000, \dots\} (\{1\}\{\varepsilon, 00, 000, \dots\})^*.$$

The generating function for  $S$  is (by the theorem on generating functions for binary strings)

$$\begin{aligned} \Phi(x) &= \frac{1 + x^2 + x^3 + \dots}{1 - x(1 + x^2 + x^3 + \dots)} \\ &= \frac{1 + \frac{x^2}{1-x}}{1 - x\left(1 + \frac{x^2}{1-x}\right)} \\ &= \frac{1 - x + x^2}{1 - 2x + x^2 - x^3}. \end{aligned}$$

The recurrence relation we get from this is

$$a_n - 2a_{n-1} + a_{n-2} - a_{n-3} = 0$$

for  $n \geq 3$  and  $a_0 = a_1 = 1$ ,  $a_2 = 2$ . The first few values of  $a_n$  are  $a_3 = 4$ ,  $a_4 = 7$ ,  $a_5 = 12$ ,  $a_6 = 21$ . Now to actually solve this recurrence equation for  $a_n$ , we have to find the roots of the characteristic equation

$$x^3 - 2x^2 + x - 1 = 0.$$

We can find the roots via computer (or using a more complicated generalization of the quadratic formula for cubic equations), and the roots turn out all to be distinct. The largest root in modulus turns out to be

$$\alpha_1 = \frac{2}{3} + \frac{1}{3} \left( \frac{25}{2} - \frac{3\sqrt{69}}{2} \right)^{\frac{1}{3}} + \frac{1}{3} \left( \frac{25}{2} + \frac{3\sqrt{69}}{2} \right)^{\frac{1}{3}} \approx 1.75488.$$

The other roots turn out to be very small in modulus (much less than one). By Theorem 1 we know therefore that

$$a_n = c_1 \alpha_1^n + c_2 \alpha_2^n + c_3 \alpha_3^n$$

where  $\alpha_1, \alpha_2, \alpha_3$  are the roots of the characteristic equation. The values of  $c_1, c_2, c_3$  are found using the initial conditions. But we can be cleverer: by the discussion on asymptotics we know that

$$a_n = [c_1 \alpha_1^n]$$

which is a much simpler formula than that given above. As an exercise, try to compute the value  $c_1$ .

### 3.3 Exercises

**Question 1.** Determine an expression which uniquely creates all strings in the given set of binary strings. Then find the generating function for that set of strings.

- (a) The set of strings which do not contain 111.
- (b) The set of strings none of whose blocks have length two.
- (c) The set of strings where every block of 0s has length at least two and every block of 1s has length at least three.

**Question 2.** Find the generating function for the set of binary strings where all blocks have the same length.

**Question 3.** Solve the following recurrence equations with the given initial conditions.

- (a)  $a_n - 13a_{n-1} + 36a_{n-2} = 0$  where  $a_0 = 0, a_1 = 5$ .
- (b)  $a_n - 2a_{n-1} + a_{n-2} = 0$  where  $a_0 = a_1 = 1$ .
- (c)  $a_n - 6a_{n-1} + 11a_{n-2} - 6a_{n-3} = 0$  where  $a_0 = 7, a_1 = 10$  and  $a_2 = 18$ .

**Question 4.** Let  $S$  be the set of binary strings consisting of a (nonempty) block of 0s followed by a (nonempty) block of 1s, such that if the block of 0s has odd length, then the block of 1s has even length. Let  $a_n$  be the number of strings of length  $n$  in  $S$ .

- (a) Find  $a_0, a_1, a_2, a_3$ .
- (b) Prove that

$$S = \{0\}\{0\}^*\{1\}\{1\}^* \setminus \{0\}\{00\}^*\{1\}\{11\}^*.$$

- (c) Show that the generating function for  $S$  is

$$\Phi_S(x) = \frac{x^3(2+x)}{(1-x^2)^2}.$$

- (d) Write down a recurrence equation for  $a_n$ .
- (e) Determine a closed formula for  $a_n$  for all  $n$ , and find  $a_{1001}$ .

**Question 5.** Write down recurrence relations with initial conditions for the quantities  $a_n$  whose generating functions  $\Phi(x) = \sum_{n=0}^{\infty} a_n x^n$  are given below. Determine the asymptotic behavior of  $a_n$  in each case.

- (a)  $\Phi(x) = \frac{1}{1-2x}$ .
- (b)  $\Phi(x) = \frac{1}{1-x-x^3}$ .
- (c)  $\Phi(x) = \frac{x^2+x+1}{x^3-3x^2+2}$ .
- (d)  $\Phi(x) = \frac{x^k-1}{x-1}$ .
- (e)\*s  $\Phi(x) = (1-x)^{1/2}$ .