

# Practice Midterm 1 – Math 154

Time: 40 Minutes | No notes allowed

Questions carry equal weight | Calculators allowed

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**Question 1.** State the Binomial Theorem, and use it to prove

$$(1 + x + x^2)^n = \sum_{k=0}^n \sum_{j=0}^k \binom{n}{k} \binom{k}{j} x^{j+k}.$$

Use the binomial theorem twice. First

$$(1 + x + x^2)^n = \sum_{k=0}^n \binom{n}{k} (x + x^2)^k = \sum_{k=0}^n \binom{n}{k} x^k (1 + x)^k$$

and then use the binomial theorem on  $(1 + x)^k$  to get the required sum.

**Question 2.** Prove by induction that

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

The question should have said just prove it directly. Whether you do that or do it by induction, the key thing to use is

$$\binom{n}{k} = \frac{n}{k} \binom{n-1}{k-1}.$$

From this it follows

$$\binom{n}{k} = \binom{n-1}{k-1} + \frac{n-k}{k} \binom{n-1}{k-1}.$$

However

$$\frac{n-k}{k} \binom{n-1}{k-1} = \frac{n-k}{k} \frac{(n-1)!}{(n-k)!(k-1)!} = \frac{(n-1)!}{(n-k-1)!k!} = \binom{n-1}{k}.$$

That does it since we then get

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

as required.

**Question 3.** Let  $n \in \mathbb{N}$ . A composition of  $n$  with  $k$  parts is a sequence  $(x_1, x_2, \dots, x_k)$  of elements of  $[n]$  such that  $x_1 + x_2 + \dots + x_k = n$ . By convention, for  $k = 0$  there is one composition, with no parts, called the empty composition. Show that there are  $\binom{n-1}{k-1}$  compositions of  $n$  with  $k$  parts for  $k \geq 1$ .

This is bookwork. The generating function is  $\Phi(x) = \left(\frac{x}{1-x}\right)^k$  which is, using the binomial theorem

$$x^k \sum_{j=0}^{\infty} \binom{-k}{j} (-x)^j.$$

To get  $[x^n]\Phi(x)$  put  $j = n - k$ . Then we get

$$[x^n]\Phi(x) = \binom{-k}{n-k} (-1)^{n-k} = \binom{n-1}{k-1}$$

since we know that  $\binom{-k}{j} = (-1)^j \binom{k+j-1}{k-1}$  (see facts on binomial coefficients).

**Question 4.** Let  $n \in \mathbb{N}$ . Determine the average number of parts in a composition of  $n$ .

Let  $S$  be the set of all compositions of  $n$  and let the weight of  $\sigma \in S$  be defined by  $\omega(\sigma) = |\sigma|$ , the number of parts in  $\sigma$ . The generating function for  $S$  is then

$$1 + \sum_{k=1}^{\infty} \binom{n-1}{k-1} x^k$$

since there is 1 composition with 0 parts and  $\binom{n-1}{k-1}$  compositions with  $k$  parts by Question 3. Using the binomial theorem

$$\Phi(x) = 1 + x \sum_{k=0}^{n-1} \binom{n-1}{k} x^k = 1 + x(1+x)^{n-1}.$$

The average number of parts is

$$\frac{\Phi'(1)}{\Phi(1)}$$

by a theorem in the notes. Clearly  $\Phi(1) = 1 + 2^{n-1}$ . Now

$$\Phi'(x) = (n-1)x(1+x)^{n-2} + (1+x)^{n-1}$$

and so

$$\Phi'(1) = (n-1)2^{n-2} + 2^{n-1}.$$

Finally, the answer to the question is

$$\frac{\Phi'(1)}{\Phi(1)} = (n-1)2^{n-2} + 2^{n-1} \cdot \frac{1}{1+2^{n-1}}.$$

You could check this formula e.g. for  $n = 3$  the compositions are the empty composition and

$$(3) \quad (2, 1) \quad (1, 2) \quad (1, 1, 1)$$

and so the average number of parts is  $(0 + 1 + 2 + 2 + 3)/5 = 8/5$  and this agrees with the formula  $(2 \cdot 2^1 + 2^2)/5 = 8/5$ .