Practice Midterm 2 Solutions – Math 154

Time: 40 Minutes | No notes allowed

Questions carry equal weight | Calculators allowed

Question 1. Let $k \in \mathbb{N}$. Find the number of walks of length k on the number line starting at zero and which return to the origin.

If k is odd, then there are no such walks since the walk has to have as many left steps as right steps. If k is even, then each walk corresponds to a sequence $(x_1, x_2, \dots, x_k)$ where $x_i \in \{L, R\}$ – here L denotes left and R denotes right – and the number of Ls and Rs is the same, both $k/2$. There are $\binom{k}{k/2}$ ways to pick where the Ls will be in the sequence, and then the rest of the sequences is Rs. Therefore there are $\binom{k}{k/2}$ walks when k is even.

Question 2. State the inclusion-exclusion formula, and use it to prove that the number of derangements of $[n]$ is $n! \sum_{k=0}^{n} \frac{(-1)^k}{k!}$.

Bookwork

Question 3. Determine closed form expressions for either one of the formal power series below:

(a) $\sum_{k=0}^{\infty} \sum_{j=0}^{k} (2+x)^{-j}(2-x)^{j-k}$.
(b) $\sum_{k=0}^{\infty} \sum_{n=0}^{k} \binom{k}{n} 2^{-n} x^{k-n}$.

(a) There are a few ways to do this. One could recall from the product rule for multiplying formal power series

$$\sum_{k=0}^{\infty} \sum_{j=0}^{k} a_j b_{k-j} = \left(\sum_{k=0}^{\infty} a_k\right)\left(\sum_{j=0}^{\infty} b_{k-j}\right)$$

provided each of the sums converges. Putting $a_j = (2+x)^{-j}$ and $b_{k-j} =$

$(2-x)^{-(k-j)}$ we get

$$\begin{aligned}
\sum_{k=0}^{\infty}\sum_{j=0}^{k}(2+x)^{-j}(2-x)^{j-k} &= \left(\sum_{k=0}^{\infty}(2+x)^{-k}\right)\left(\sum_{j=0}^{\infty}(2-x)^{-j}\right) \\
&= \frac{1}{1-\frac{1}{2+x}}\cdot\frac{1}{1-\frac{1}{2-x}} \qquad \text{geometric series} \\
&= \frac{(2+x)(2-x)}{(2+x-1)(2-x-1)} \\
&= \frac{4-x^2}{1-x^2}.
\end{aligned}$$

(b) This would not be examinable since have not dealt with the formal power series e^x. Interchanging the order of summation, we get

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=n}^{\infty}\binom{k}{n}2^{-n}x^{k-n} &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\binom{k+n}{n}2^{-n}x^{k} \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2^n n!}\cdot(k+n)(k+n-1)\cdots(k+1)x^k.
\end{aligned}$$

Now

$$\frac{d^n}{dx^n}x^{k+n} = (k+n)(k+n-1)\cdots(k+1)x^k.$$

Therefore

$$\begin{aligned}
\sum_{n=0}^{\infty}\sum_{k=0}^{\infty}\frac{1}{2^n n!} &\cdot \quad (k+n)(k+n-1)\cdots(k+1)x^k \\
&= \sum_{n=0}^{\infty}\frac{1}{2^n n!}\frac{d^n}{dx^n}\sum_{k=0}^{\infty}x^{k+n} \\
&= \sum_{n=0}^{\infty}\frac{x^n}{(1-x)2^n n!} \qquad \text{geometric series} \\
&= \frac{1}{1-x}\sum_{n=0}^{\infty}\frac{1}{n!}\left(\frac{x}{2}\right)^n \\
&= \frac{e^{x/2}}{1-x}.
\end{aligned}$$

Question 4. Find the number of compositions of n in which each part is odd.

Bookwork – see notes for this example.