

# Practice Midterm 3 Solutions – Math 154

Time: 40 Minutes | No notes allowed

Questions carry equal weight | Calculators allowed

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**Question 1.** Let  $e \approx 2.718\dots$  denote the base of the natural logarithm. Prove by induction on  $n$  that

$$n! \geq n^n e^{-n}.$$

You may use the fact that

$$\left(1 - \frac{1}{n}\right)^{n-1} > \frac{1}{e}.$$

For  $n = 1$  we get  $1 = 1! \geq 1/e$  which is true, so the base case works. Suppose that  $(n-1)! \geq (n-1)^{n-1} e^{-(n-1)}$ . Using this we wish to prove that  $n! \geq n^n e^{-n}$ . Then

$$n! = n(n-1)! \geq n(n-1)^{n-1} e^{-(n-1)} = n^n \left(1 - \frac{1}{n}\right)^{n-1} e^{-n} \cdot e.$$

By the given fact,

$$\left(1 - \frac{1}{n}\right)^{n-1} > \frac{1}{e}$$

so putting this in the preceding equation we get

$$n! > n^n \frac{1}{e} e^{-n} \cdot e = n^n e^{-n}$$

as required.

**Question 2.** Determine the generating function for the set  $S$  of positive integers with respect to the weight function  $\omega(\sigma) = \lfloor \log_2 \sigma \rfloor$ . Here  $\lfloor x \rfloor$  denotes the largest integer less than or equal to  $x$ .

We know that

$$\Phi_S(x) = \sum_{k=0}^{\infty} a_k x^k$$

where  $a_k$  is the number of things in  $S$  of weight  $k$ . In other words,  $a_k$  is the number of positive integers  $\sigma$  such that  $\omega(\sigma) = \lfloor \log_2 \sigma \rfloor = k$ . So it follows that  $a_0 = 1$  since only 1 has  $\omega(1) = \log_2(1) = 0$ . Now  $a_1 = 2$  since  $\omega(2) = \omega(3) = 1$ .

In general,  $a_k = 2^k$  since all the integers from  $2^k$  to  $2^{k+1} - 1$  have weight  $k$ . Therefore

$$\Phi_S(x) = \sum_{k=0}^{\infty} 2^k x^k = \frac{1}{1-2x}$$

and in the last step we recognised the sum as a geometric series.

**Question 3.** Determine explicitly the value of the sum

$$\sum_{k=0}^{\infty} \sum_{j=0}^k \binom{1/2}{j} \binom{1/2}{k-j} 2^{2k+j}.$$

Using the product rule for formal power series, which is

$$\sum_{k=0}^{\infty} \sum_{j=0}^k a_j b_{k-j} x^k = \sum_{k=0}^{\infty} a_k x^k \cdot \sum_{k=0}^{\infty} b_k x^k$$

with  $a_j = \binom{1/2}{j} 2^j$  and  $b_{k-j} = \binom{1/2}{k-j}$  and  $x = 4$  we get that the sum is

$$\sum_{k=0}^{\infty} \binom{1/2}{k} 8^k \cdot \sum_{k=0}^{\infty} \binom{1/2}{k} 4^k = (1+8)^{1/2} \cdot (1+4)^{1/2} = 3\sqrt{5}$$

where we used the binomial theorem in the last step.

**Question 4.** The binary strings which start with a 1 and end with a 0 and do not contain 010 or 101 are uniquely created by the formula

$$S = \{1\}\{1\}^*\{00\}\{0\}^*\{11\}\{1\}^*\{0\}\{0\}^*.$$

- If  $a_n$  is the number of strings of length  $n$  in  $S$ , find  $a_n$  for  $n = 0, 1, 2, 3, 4$ .
- Write down the generating function for these strings.
- Determine a recurrence equation for  $a_n$ , the number of strings of length  $n$  in  $S$ .

(a) There are no strings of length zero or one starting with 1 and ending with 0 so  $a_0 = a_1 = 0$ . Now  $a_2 = 1$  since only 10 starts with a 1 and ends with a zero. Next,  $a_3 = 2$  since the only valid strings are 110 and 100. Finally  $a_4 = 3$  since the strings are 1110, 1100, 1000.

(b) From the expression given we get

$$\Phi_S(x) = \frac{x^2}{(1-x)^2} \left( \frac{1}{1 - \frac{x^4}{(1-x)^2}} \right) = \frac{x^2}{(1-x)^2 - x^4} = \frac{x^2}{1 - 2x + x^2 - x^4}.$$

(c) From the generating function we get  $a_n = 2a_{n-1} + a_{n-2} - a_{n-4}$ .