

Practice Midterm 4 Solutions – Math 154

Time: 40 Minutes | No notes allowed

Questions carry equal weight | Calculators allowed

Question 1. Let \mathcal{S} be the set of subsets of $[2n] := \{1, 2, \dots, 2n\}$ in which every element is even, and let \mathcal{T} be the set of subsets of $[2n]$ in which every element is less than $n + 1$. Find a bijection $f : \mathcal{S} \leftrightarrow \mathcal{T}$.

For each set $A \in \mathcal{S}$, define

$$f(A) = \left\{ \frac{a}{2} : a \in A \right\}.$$

Then $f(A) \in \mathcal{T}$ for all $A \in \mathcal{S}$ since $\max f(A) \leq \frac{2n}{2} = n < n + 1$. The function for $B \in \mathcal{T}$ defined by

$$g(B) = \{2b : b \in B\}$$

is clearly the inverse of f , and therefore f is a bijection.

Question 2. State the inclusion-exclusion formula. Use the formula to determine how many integers in $\{1, 2, \dots, 100\}$ are divisible by a single digit prime number?

The single digit primes are 2, 3, 5, 7. We want the numbers up to 100 divisible by one of these numbers, so let A_p be the number of integers up to 100 divisible by p . Then by inclusion-exclusion

$$\begin{aligned} |A_2 \cup A_3 \cup A_5 \cup A_7| &= |A_2| + |A_3| + |A_5| + |A_7| \\ &- |A_2 \cap A_3| - |A_2 \cap A_5| - |A_2 \cap A_7| - |A_3 \cap A_5| - |A_3 \cap A_7| - |A_5 \cap A_7| \\ &+ |A_2 \cap A_3 \cap A_5| + |A_2 \cap A_3 \cap A_7| + |A_3 \cap A_5 \cap A_7| + |A_2 \cap A_5 \cap A_7| \\ &- |A_2 \cap A_3 \cap A_5 \cap A_7|. \end{aligned}$$

Now $|A_p| = \lfloor \frac{100}{p} \rfloor$ so $|A_2| = 50$, $|A_3| = 33$, $|A_5| = 20$, $|A_7| = 14$. Also $|A_p \cap A_q| = \lfloor \frac{100}{pq} \rfloor$ when p and q are different primes so $|A_2 \cap A_3| = 16$, $|A_2 \cap A_5| = 10$, $|A_2 \cap A_7| = 7$, $|A_3 \cap A_5| = 6$, $|A_3 \cap A_7| = 4$, $|A_5 \cap A_7| = 2$. Also $|A_2 \cap A_3 \cap A_5| = 3$, $|A_2 \cap A_3 \cap A_7| = 2$, $|A_2 \cap A_5 \cap A_7| = 1$, and $|A_3 \cap A_5 \cap A_7| = 0 = |A_2 \cap A_3 \cap A_5 \cap A_7|$. So the answer is

$$50 + 33 + 20 + 14 - 16 - 10 - 7 - 6 - 4 - 2 + 3 + 2 + 1 = 78.$$

For your information the numbers are all even numbers up to 100 together with

3, 5, 7, 9, 15, 21, 25, 27, 33, 35, 39, 45, 49, 51, 55, 57, 63, 65, 69, 75, 77, 81, 85, 87, 91, 93, 95, 99.

Question 3. Solve the recurrence equation $a_n = a_{n-1} + 2a_{n-2}$ with initial conditions $a_0 = a_1 = 1$.

The characteristic equation is $x^2 - x - 2 = 0$ with roots $x = 2$ and $x = -1$. Therefore

$$a_n = c_1 2^n + c_2 (-1)^n$$

where c_1, c_2 are constants. Now

$$a_0 = c_1 + c_2 = 1 \quad a_1 = 2c_1 - c_2 = 1 \quad \Rightarrow \quad c_1 = \frac{2}{3} \text{ and } c_2 = \frac{1}{3}.$$

So

$$a_n = \frac{2}{3} 2^n + \frac{1}{3} (-1)^n.$$

Question 4. Find the number of binary strings of length n not containing 011 or 001.

Let S be the set of strings not containing 011 or 001. Then the strings in S are uniquely created by

$$S = \{1\}^* (\{0\}\{1\})^* \{0\}^*.$$

So the generating function is

$$\Phi_S(x) = \frac{1}{(1-x)^2} \frac{1}{1-x^2} = \frac{1}{1-2x+2x^3-x^4}.$$

If a_n is the number of strings of length n in S then

$$a_n = 2a_{n-1} - 2a_{n-3} + a_{n-4}.$$

We also know $a_0 = 1$, $a_1 = 2$, $a_2 = 4$ and $a_3 = 6$ since the strings of length three which are allowed are 000, 010, 110, 101, 111, 100. The characteristic equation is

$$\alpha^4 - 2\alpha^3 + 2\alpha - 1 = 0.$$

This factorizes into (check $\alpha = 1$ is a root repeatedly and $\alpha = -1$ is a root)

$$(\alpha - 1)^3(\alpha + 1) = 0.$$

Therefore

$$a_n = (an^2 + bn + c)1^n + d(-1)^n$$

for some constants a, b, c, d . From the initial conditions we get

$$c + d = 1$$

$$a + b + c - d = 2$$

$$4a + 2b + c + d = 4$$

$$9a + 3b + c - d = 6$$

Using linear algebra we get $a = \frac{1}{4}$, $b = 1$, $c = \frac{7}{8}$, $\frac{1}{8}$ and therefore

$$a_n = \frac{1}{4}n^2 + n + \frac{7}{8} + \frac{1}{8}(-1)^n.$$