Practice Final Examination #2

Math 154 – Combinatorics and Graph Theory
Instructor – J. Verstraete
Allotted time – 3 hours

Answers are to be written clearly and legibly
Calculators are allowed
State clearly any theorems used without proof
Total 50 points
Question 1.

(a) Let $d \in \mathbb{N}$. Prove that the generating function for the number of compositions of $n$ where each part is a multiple of $d$ is

$$
\Phi(x) = \frac{1-x^d}{1-2x^d}.
$$

(b) Prove that there are $2^{n/d-1}$ such compositions if $n$ is a multiple of $d$.

Number of such compositions into $k$ parts:

$$
S(k) = S_1 \times S_2 \times \cdots \times S_k.
$$

$$
S_\ell = \{ d, 2d, 3d, \ldots \}.
$$

$$
\phi_{S_\ell}(x) = x^d + x^{2d} + x^{3d} + \ldots = \frac{x^d}{1-x^d}.
$$

So

$$
\frac{d(x)}{S(k)} = \left( \frac{x^d}{1-x^d} \right)^k \quad \text{PRODUCT LEMMA}
$$

$$
S = \text{all compositions} = \bigcup_{k=0}^\infty S(k).
$$

$$
\phi_{\bigcup_{k=0}^\infty S(k)}(x) = \sum_{k=0}^\infty \left( \frac{x^d}{1-x^d} \right)^k = \frac{1}{1-x^d}.
$$

Geometric Series

$$
\frac{1-x^d}{1-2x^d}.
$$

(b) $[x^n] \phi_{S}(x) = [x^n] \left( \frac{1-x^d}{1-2x^d} \right) = [x^n] (1-x^d) \cdot \sum_{j=0}^{\infty} \left( 2^{\lfloor d \rfloor} x \right)^j = 2^{n/d} - 2^{n/d-\lfloor d \rfloor} \quad \text{put } j=0
$$

$$
= 2^{n/d} - 1 \quad \text{THEN put j=n- \lfloor d \rfloor}
$$
Question 2.

(a) Let $S$ be the set of all binary strings not containing $110$. Write down all the strings of length 1, 2 and 3 in $S$.

(b) Choose the expression uniquely creates all binary strings in $S$.

1. $\{0\}^*\{(1)^*\{0\}^*\{1\}^*$
2. $\{0\}^*\{(1)\{0\}\{0\}^*\{1\}^*$
3. $\{0\}^*\{(1)^*\{0\}^*\{1\}^*$

(c) Write down the generating function for the number of strings of length $n$ in $S$.

(d) If $a_n$ is the number of strings of length $n$ in $S$, find a recurrence equation for $a_n$.

(e) Find $\lim_{n \to \infty} a_n^{1/n}$.

(a) 

\begin{align*}
0, & \text{ length 1} \\
00, 01, 10, 11, & \text{ length 2} \\
000, 001, 100, 101, 111, 010, 011, & \text{ length 3}
\end{align*}

(b) It is (2).

(c) From (2):

\[
\phi_S(x) = \frac{1}{(1-x)^2} \cdot \frac{1}{1-x^2} = \frac{1}{(1-2x+x^2) - x^2(1-x)} = \frac{1}{1-2x+x^3}
\]

(d) $a_n = 2a_{n-1} + a_{n-3}$

(e) Characteristic equation $x^3 - 2x^2 + 1 = 0$

$x = 1$ is a solution, so $(x-1)(x^2-x-1) = 0$

and so we get $x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}$

\[
\lim_{n \to \infty} a_n^{1/n} = \frac{\sqrt{5} + 1}{2}
\]
Question 3.

(a) State and prove the handshaking lemma. [3]
(b) Prove that a tree on $n$ vertices has $n - 1$ edges. [4]
(c) Show that every tree has at least two vertices of degree one. [2]
(d)* Let $T$ and $U$ be edge-disjoint binary trees on a set $V$ of vertices. Prove that if $G = T \cup U$ then $\lambda(G) = \delta(G) = 2$. Is $\kappa(G) = 2$? State any theorems used without proof. [3]

(a) (b) Bookwork

(c) By (a) and (b), if $T$ is a tree then

$$\sum_{v \in T} d(v) = 2(n-1)$$

when $T$ has $n$ vertices.

If $d(v) \geq 2$ always then

$$\sum_{v \in T} d(v) \geq 2n \quad \text{a contradiction}.$$ 

If $d(v) \geq 2$ for all but one vertex, then that vertex has degree $\geq 1$ so

$$\sum_{v \in T} d(v) \geq 2(n-1)+1 = 2n-1$$

and another contradiction.
Question 3...

(d) Example of a binary tree.

Number of leaves in a binary tree on \( n \) vertices is \( \geq \frac{n}{2} \).

So two binary trees on \( n \) vertices must have a leaf in common.

Thus \( \delta(G) = 2 \).

\( \lambda(G) \leq \delta(G) = 2. \)

Also \( \lambda(G) = 2 \) since any two vertices are joined by 2 edge-disjoint paths: one in \( T \) and one in \( U. \) Menger’s Theorem gives \( \lambda(u,v) \geq 2 \) and so \( \lambda(G) \geq \lambda(u,v) \geq 2. \)
Question 4.

(a) Define \( \Gamma(X) \) when \( X \subseteq A \) is a set of vertices of a bipartite graph \( G = (A \cup B, E) \).

(b) State Hall's Theorem.

(c) Let \( n \in \mathbb{N} \). Prove that in a bipartite graph \( G = (A \cup B, E) \) with \( \delta(G) \geq n \) and \( |A| = |B| = 2n \), there is a perfect matching using Hall's Theorem.

\[ \begin{align*}
(a, b) & \quad \text{bookwork} \\
(c) & \quad \text{We prove } |\Gamma(X)| \geq |X| \text{ for all } X \subseteq A \text{ and } X \subseteq B. \\
\text{If } |X| \leq n, \text{ then } |\Gamma(X)| \geq n \text{ since } \delta(G) \geq n \text{ and so in this case } |\Gamma(X)| \geq |X|. \\
\text{If } |X| > n, \text{ then for } X \subseteq A \Gamma(X) = B. \\
\text{since } \delta(G) \geq n, \text{ for any } b \in B, b \text{ has a neighbour in } X \text{ since } |A \setminus X| < n \text{ and } d(b) \geq n. \text{ Same for } X \subseteq B. \\
\text{So } |\Gamma(X)| \geq |X| \text{ for all } X \\
\Rightarrow \text{perfect matching by Hall's Theorem.}
\end{align*} \]
Question 4...
Question 5.

Find a maximum $st$-flow and minimum $st$-cut in the network shown below. Show all working.

![Network Diagram]

Figure 1: A network
Question 5...