Midterm Examination

Math 154 – Combinatorics and Graph Theory

Instructor – J. Verstraete

Allotted time – 40 minutes

State clearly any theorems used without proof
All questions carry a weight of ten points

Question 1.

In this question, n and k are positive integers. Prove the following by any method:

(a) $\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1}$. [3]

(b) $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$. [7]

(a) By definition,

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \checkmark$$
$$= \frac{n\cdot(n-1)!}{k(k-1)!(n-k)!} \checkmark$$
$$= \frac{n}{k}\frac{(n-1)!}{(k-1)!(n-k)!} = \frac{n}{k}\binom{n-1}{k-1} \checkmark$$

(b) Using (a),

$$\binom{n}{k} = \frac{n}{k}\binom{n-1}{k-1} \checkmark$$
$$= \frac{n-k}{k}\binom{n-1}{k-1} + \frac{k}{k}\binom{n-1}{k-1} \checkmark \checkmark$$
$$= \frac{n-k}{k}\cdot\frac{(n-1)!}{(k-1)!(n-k)!} + \frac{k}{k}\binom{n-1}{k-1}$$
$$= \frac{(n-1)!}{k!(n-k-1)!} \checkmark + \binom{n-1}{k-1} \checkmark$$
$$= \binom{n-1}{k} + \binom{n-1}{k-1} \checkmark\checkmark$$

Question 2.

(a) State the inclusion-exclusion formula. [3]
(b) Complete the expression $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| \cdots$ [3]
(c) How many positive integers less than 50 are not divisible by 2, 3 or 7? [4]

(a) Let $A_1, A_2, \ldots, A_n$ be finite sets. Then

$$\left|\bigcup_{i=1}^{n} A_i\right| = \sum_{\phi \neq S \subseteq [n]} (-1)^{|S|+1} \left|\bigcap_{i \in S} A_i\right|$$

(b) $$|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$$

(c) Let A_1, A_2 and A_3 be the sets of integers less than 50 divisible by 2, 3 and 7 respectively.
So $|A_1| = 24$, $|A_2| = 16$, $|A_3| = 7$

Then we want

$$49 - |A_1 \cup A_2 \cup A_3|.$$

By inclusion-exclusion (ie. (b))

$$|A_1 \cup A_2 \cup A_3| = 24 + 16 + 7 - 8 - 3 - 2 + 1 = 35$$

$\Longrightarrow$ Therefore there are 14 such integers. They are $\{1, 5, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47\}$

Question 3.

Let S be the set of binary strings not containing 110 or 011 or 000 and which start with a 0. Let a_n be the number of strings of length n in S.

(a) Which one of the following expressions uniquely creates all the strings in S? [2]
  (1) $\{0,00\}(\{1\}\{1\}^*\{0\})^*\{1\}\{1\}^*$
  (2) $(\{0,00\}\{1\}\{1\}^*)^*$
  (3) $\{0,00\}(\{1\}\{0,00\})^*\{\varepsilon,1\}$

(b) Write down the generating function for each of the sets of strings in (a). [5]

(c) Write down a recurrence equation for a_n and find a_{10}. [3]

(a) (3) Since no block of 1s after an initial 0 can have length 2 or more, the double blocks must be in $\{1\}\{0\}\{0\}^*$, and the end must be $\{\epsilon, 1\}$ (otherwise we get 011). ✓✓

(b) (1) $\dfrac{(x+x^2)x}{1-x} \cdot \dfrac{1}{1-\dfrac{x^2}{1-x}}$ ✓

(2) $\dfrac{1}{1-\dfrac{x^2+x^3}{1-x}}$ ✓

(3) $\dfrac{(x+x^2)(1+x)}{1-(x^2+x^3)} = \dfrac{x(1+x)^2}{1-x^2-x^3}$ ✓

(c) From (b)(3): $a_n = a_{n-2} + a_{n-3}$. ✓

$$\begin{aligned}
a_{10} &= a_8 + a_7 \\
&= a_5 + a_6 + a_4 + a_5 \\
&= a_2 + a_3 + a_3 + a_4 + a_1 + a_2 + a_2 + a_3 \\
&= 3a_2 + 3a_3 + a_4 + a_1 \\
&= 3a_2 + 3a_3 + a_1 + a_2 + a_1 \\
&= 4a_2 + 3a_3 + 2a_1 \quad ✓
\end{aligned}$$

$a_3 = 2$ (all except start with 0 110, 011, 000 so 001 and 010) ✓

$a_2 = 2$ (01 and 00) ✓

$a_1 = 1$ (just 0)

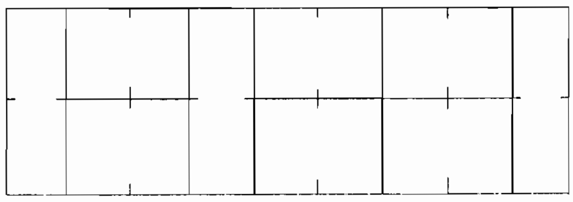
so $a_{10} = 4\cdot 2 + 3\cdot 2 + 2\cdot 1$

$= 15.$ ✓

Question 4.

(a) Solve the <u>Fibonacci recurrence</u> equation $a_n = a_{n-1} + a_{n-2}$, with initial conditions $a_0 = 0$ and $a_1 = 1$. [7]

(b) The 2×9 board is shown below and is tiled with grey dominoes. Show that the number of ways of tiling the board with dominoes is 55. [3]



(a) Characteristic equation

$$x^2 - x - 1 = 0$$

roots $x = \frac{1 \pm \sqrt{5}}{2}$ with multiplicity 1 each

therefore

$$a_n = c_1 \left(\frac{1+\sqrt{5}}{2} \right)^n + c_2 \left(\frac{1-\sqrt{5}}{2} \right)^n$$

$a_0 = 0 \Rightarrow c_1 + c_2 = 0$

$a_1 = 1 \Rightarrow c_1 \left(\frac{1+\sqrt{5}}{2} \right) + c_2 \left(\frac{1-\sqrt{5}}{2} \right) = 1$

So $c_1 = -c_2$ and in the second equation we get

$$c_1 \sqrt{5} = 1$$

so $c_1 = \frac{1}{\sqrt{5}} \rightarrow c_2 = -\frac{1}{\sqrt{5}}$

and

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^n.$$

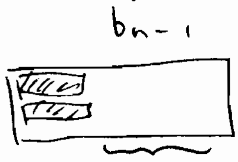
(b) For the $2 \times n$ board, let b_n be the number of tilings. Then

$$b_n = b_{n-1} + b_{n-2}$$ and $b_1 = 1$, $b_2 = 2$

b_{n-1}: a domino at the end is vertical

b_{n-2}: two horizontal dominoes at the end

b_{n-1}



b_{n-2}

Hence from (a)

$$b_9 = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}$$

$$= 55.$$

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