Question 1.

In this question, \( n \) and \( k \) are positive integers. Prove the following by any method:

(a) \( \binom{n}{k} = \frac{n(n-1)}{k(k-1)} \)

(b) \( \binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1} \)

(a) By definition,

\[
\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)!}{k(k-1)!(n-k)!} = \frac{n}{k} \frac{(n-1)!}{(k-1)!(n-k)!} = \binom{n}{k} \frac{(n-1)}{k(k-1)}
\]

(b) Using (a),

\[
\binom{n}{k} = \frac{n}{k} \frac{(n-1)}{k-1} = \frac{n-k}{k} \frac{(n-1)}{k-1} + \frac{k}{k} \frac{(n-1)}{k-1} = \frac{(n-1)}{k} \frac{(n-1)!}{(k-1)!(n-k)!} + \frac{(n-1)}{k-1} = \binom{n-1}{k-1}
\]
Question 2.

(a) Let $A_1, A_2, ..., A_n$ be finite sets. Then
\[
\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{\emptyset \neq S \subseteq \{1, 2, ..., n\}} (-1)^{|S|+1} \left| \bigcap_{i \in S} A_i \right| \checkmark
\]

(b) \[
|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3| \checkmark
\]

(c) Let $A_1, A_2$ and $A_3$ be the sets of integers less than 50 divisible by 2, 3 and 7 respectively. So
\[
|A_1| = 24, \quad |A_2| = 16, \quad |A_3| = 7
\]

Then we want
\[
49 - |A_1 \cup A_2 \cup A_3| \checkmark
\]

By inclusion-exclusion (i.e. \((b)\))
\[
|A_1 \cup A_2 \cup A_3| = 24 + 16 + 7 - 8 - 3 - 2 + 1 = 35 \checkmark
\]

Therefore there are 14 such integers. They are \{15, 11, 13, 17, 19, 23, 25, 29, 31, 37, 41, 43, 47\}
Question 3.

Let $S$ be the set of binary strings not containing 110 or 011 or 000 and which start with a 0. Let $a_n$ be the number of strings of length $n$ in $S$.

(a) Which one of the following expressions uniquely creates all the strings in $S$? [2]

(1) $\{0,0\}{\{1\}^*}^*\{0\}^*{\{1\}^*}$
(2) $\{0,0\}{\{1\}^*}^*$
(3) $\{0,0\}{\{1\}^*}\{0,0\}$

(b) Write down the generating function for each of the sets of strings in (a). [5]

(c) Write down a recurrence equation for $a_n$ and find $a_{10}$. [3]

(a) (3) Since no block of 1s after an initial 0 can have length 2 or more, the double block must be $1^*0^*0^*0^*$ and the end must be $\{0,1\}$ (otherwise we get 011).

(b) (1) $$\frac{(x+x^2)x}{1-x} \cdot \frac{1}{1-x^2} = \frac{x}{1-x^2-x^3}$$

(2) $$\frac{1}{1-x} = \frac{1}{1-x^2}$$(3) $$\frac{x(1+y)^2}{x+xy}$$

(c) From (b)(3): $a_n = a_{n-2} + a_{n-3}$.

$a_{10} = a_8 + a_7$

$= a_5 + a_6 + a_5 + a_6$

$= a_2 + a_3 + a_4 + a_5 + a_3 + a_4 + a_2 + a_3$

$= 3a_2 + 3a_3 + a_4 + a_1$

$= 3a_2 + 3a_3 + a_4 + a_2 + a_1$

$= 4a_2 + 3a_3 + 2a_1$

$a_3 = 2$ (with 0

$\text{short with } 0$ (all except 110, 011, 000) * 001 and 010)

$a_2 = 2$ (01 and 00)

$a_1 = 1$ (with 0)

so $a_{10} = 4 \cdot 2 + 3 \cdot 2 + 2 \cdot 1$

$= 15$. √
Question 4.

(a) Solve the Fibonacci recurrence equation $a_n = a_{n-1} + a_{n-2}$, with initial conditions $a_0 = 0$ and $a_1 = 1$.

(b) The $2 \times 9$ board is shown below and is tiled with grey dominoes. Show that the number of ways of tiling the board with dominoes is 55.

(a) Characteristic equation

$$x^2 - x - 1 = 0$$

with multiplicity 1 each.

Therefore

$$a_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n$$

$$a_0 = 0 \implies c_1 + c_2 = 0$$

$$a_1 = 1 \implies c_1 \left( \frac{1 + \sqrt{5}}{2} \right) + c_2 \left( \frac{1 - \sqrt{5}}{2} \right) = 1$$

So $c_1 = -c_2$ and in the second equation we get

$$c_1 \sqrt{5} = 1$$

so

$$c_1 = \frac{1}{\sqrt{5}} \implies c_2 = -\frac{1}{\sqrt{5}}$$

and

$$a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^n.$$