2.14 Exercises

Question 1. Determine the exact numerical values of the following binomial coefficients:

(a) $\binom{1}{2}$
(b) $\binom{2}{0}$
(c) $\binom{n}{n-1}$
(d) $\binom{1/2}{4}$
(e) $\binom{-1/2}{4}$
(f) $\binom{-1/3}{3}$
(g) $\binom{1/3}{k}$.

Question 2. Find the inverses of the following formal power series as a sum of powers of x, or state that the power series has no inverse. Justify your answers.

(a) $1 - 2x + x^2$
(b) $x(1 - x)$
(c) $4 - x^2$
(d) $1 - x + x^2 - x^3 + x^4 - x^5 + \ldots$
(e) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots$
(f) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$.

Question 3. Determine the generating function in closed form for the given set S of configurations with weight function ω.

(a) S is the set of subsets of $[n]$, $\omega(\sigma) = |\sigma|$
(b) $S = \mathbb{Z}^+$ and $\omega(\sigma) = 2\sigma$.
(c) $S = \mathbb{Z}^+$ and $\omega(\sigma) = \sigma$ if σ is odd and $\omega(\sigma) = 0$ otherwise.
(d) $S = [n]$ and $\omega(\sigma) = 1$ if σ is odd and $\omega(\sigma) = 0$ if σ is even.
(e) S is the set of permutations of $[4]$, $\omega(\sigma)$ is the number of fixed points of σ.
(f) S is the set of pairs (a, b) of positive integers, and $\omega(a, b) = a + b$.
(g) S is the set of pairs (a, b) of positive integers, and $\omega(a, b) = 2a + b$.

Question 4. Compute the average maximum element of non-empty subsets of $[n]$.

Question 5. What is the average difference between the largest and smallest elements of a non-empty subset of $[n]$?

Question 6. Determine the number of compositions of n into k parts with the given restrictions.

(a) Each part is a positive even integer.
(b) Each part is an element of $\{2, 3\}$
(c) The ith part is at most i
(d) Exactly one part is odd, the rest are positive and even.

Question 7. Determine the number of compositions of n into any number of parts, where each part is odd.

Question 8. How many sets $\{x_1, x_2, \ldots, x_k\} \subset [n]$ have $x_{i-1} + i \leq x_i$ for $2 \leq i \leq k$?

Question 9* Let S denote the set of subsets of positive integers of size k, with weight function $\omega(\{x_1, x_2, \ldots, x_k\}) = x_1 + x_2 + \ldots + x_k$. Prove that

$$\Phi_S(x) = \prod_{r=1}^{\infty} \frac{1}{1 - x^r}.$$

Question 10. Determine the number of binary strings of length n with the given restrictions.

(a) The strings have only blocks of odd length.
(b) The strings do not contain 011.
(c) The strings do not contain 101 or 010.
(d) The strings do not contain 101.

Question 11. Determine the generating function for the set of binary strings of all lengths not containing 011.

Question 12. Determine a formula for the generating function of the set of k-ary trees: these are trees with a root vertex and in which every vertex has at most k children.