Question 1. Determine the number of sequences $(x_1, x_2, \ldots, x_k)$ where $x_i \in [n]$ with the given restrictions.

(a) For $i \leq n$, x_i is odd.
(b) For $i \leq n$, $x_i \neq x_{i+1}$.
(c) For $i \leq n$, $x_{i+1} > x_i$.
(d) For $i \leq n$, $x_{i+1} > x_i + 1$.

Solution (a) There are $\lceil n/2 \rceil$ choices for each x_i since there are $\lceil n/2 \rceil$ odd numbers in $[n]$. So the multiplication principle says that there must be $\lceil n/2 \rceil^k$ sequences.

Solution (b) There are n choices for x_1, and thereafter $n-1$ choices for the remaining entries (at each stage pick an entry different to the preceding one). So the multiplication principle says the answer is $n(n-1)^{k-1}$.

Solution (c) For each sequence $(x_1, x_2, \ldots, x_k)$ with $x_i < x_{i+1}$, form the set $\{x_1, x_2, \ldots, x_k\}$. Then the function

$$f(x_1, x_2, \ldots, x_k) = \{x_1, x_2, \ldots, x_k\}$$

is a bijection because for each set $A \subseteq [n]$ of size k we can form the increasing sequence a of its elements and then $f(a) = A$ (the function is onto) and each sequence σ clearly gives a unique set $f(\sigma)$. Therefore there are $\binom{n}{k}$ sequences in this case.

Solution (d) For each sequence $(x_1, x_2, \ldots, x_k)$ with $x_i + 1 < x_{i+1}$, form the set $\{x_1, x_2 - 1, \ldots, x_k - k + 1\}$. Then the function

$$f(x_1, x_2, \ldots, x_k) = \{x_1, x_2 - 1, \ldots, x_k - k + 1\}$$

is a bijection between the sequences we are trying to count and the subsets of $[n-k+1]$ of size k – note that $x_k \in [n]$ means $x_k \in [x_k - k + 1]$. To check that f is onto, note that if $\{y_1, y_2, \ldots, y_k\}$ is a set in $[n-k+1]$ and $y_1 < y_2 < \ldots < y_k$, then

$$f(y_1, y_2 + 1, y_3 + 2, \ldots, y_k + k - 1) = \{y_1, y_2, \ldots, y_k\}.$$

As in part (c), f is one-to-one. So we have the required bijection, and the number of sequences we want must be $\binom{n-k+1}{k}$.

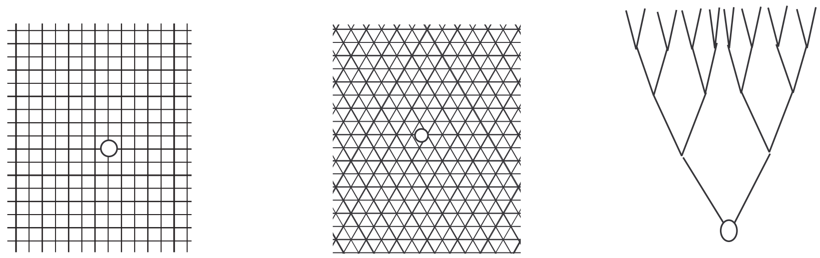

Figure 4 : Square and triangular lattices, and infinite binary tree.

Question 2. Consider the portions of the infinite lattices shown above.

(a) Determine the number of walks of length k which start at the circled point.
(b) Determine the number of walks of length k which start at the circled point but which never traverse the same edge twice in consecutive steps.
(c)* Determine the number of walks of length k which start and end at the circled point.

Solution (a) There are four choices at each point of the square grid and six choices for the triangular grid, so the answer in those cases is 4^k and 6^k, respectively. We do not do the third picture for part (a).

Solution (b) For the square grid, there are 4 choices at the first step of the walk, but then 3 choices thereafter. So the answer is $4 \cdot 3^{k-1}$. Similarly in the triangular grid the answer is $6 \cdot 5^{k-1}$. In the binary tree, the answer is 2^k since there are two choices at each step because we always have to step upwards in the tree.

Solution (c)* On the square grid, to come back to the origin, the number of steps, k, must be even. every walk of length k corresponds to a sequence $(x_1, x_2, \dots, x_k)$, where $x_i \in (L, R, U, D)$ and L, R, U, D correspond to the direction taken. So if $x_i = L$, for example, that means we took one step left in the ith step of the walk. Now if the walk returns to the origin, then in the sequence $(x_1, x_2, \dots, x_k)$, there are the same number of Ls as Rs, and the same number of Ds as Us. Call such sequences balanced. The number of such sequences is exactly

$$\binom{k}{k/2}^2.$$

We can see this as follows, using bijections. Clearly $\binom{k}{k/2}^2$ is the number of pairs (A, B) such that $A, B \subseteq [k]$ and $|A| = |B| = k/2$. Let X be the set of these pairs. Let Y be the set of balanced sequences of length k, and define the function $f : X \to Y$ as follows: define $f(A, B) = (x_1, x_2, \dots, x_k)$ where

$$x_i = \begin{cases} L & \text{for } i \in A \cap B \\ U & \text{for } i \in A \backslash B \\ D & \text{for } i \in B \backslash A \\ R & \text{otherwise} \end{cases}$$

To see that f is a bijection, note that $A = \{i : x_i = L \text{ or } x_i = U\}$ and $B = \{i : x_i = D \text{ or } x_i = R\}$, so for every $(x_1, x_2, \ldots, x_k)$ we can uniquely recover (A, B). So f is a bijection. This proves what we wanted for the square grid.

For the triangular grid, we have in a similar way a sequence $(x_1, x_2, \ldots, x_k)$ where k is a multiple of three and $x_i \in \{L, R, DL, UL, DR, UR\}$ where DL means the step was down and to the left, UR means up and to the right, and so on, and the number of Ls equals the number of Rs, the number of DLs equals the number of URs, and the number of ULs equals the number of DRs. If we choose first ℓ positions where L, R, DL, UR will be, then we are back to the above argument and the number of placements of L, R, DL, UR in the ℓ positions is $\binom{\ell}{\ell/2}^2$. There are $\binom{k}{\ell}$ choices for those positions, and then $\binom{k-\ell}{(k-\ell)/2}$ choices for how to split the remaining positions into DR and UL. So the answer is

$$\sum_{\ell=0}^{k} \binom{k}{\ell} \binom{\ell}{\ell/2}^2 \binom{k-\ell}{(k-\ell)/2}.$$

With a bit of manipulation this is

$$\binom{k}{k/2} \sum_{\ell=0}^{k/2} \binom{k/2}{\ell/2} \binom{\ell}{\ell/2}.$$

For the last lattice, we have a sequence $(x_1, x_2, \ldots, x_k)$ where k is even and $x_i \in \{LU, RU, D\}$ where L, R, U, D stand for left right up and down respectively, and where the number of LU steps plus the number of RU steps equals the number of D steps. That means there are $k/2$ of the D steps, and then the remaining $k/2$ steps can be partitioned in any way into LU and RU steps. It follows that the answer is $2^{k/2}\binom{k}{k/2}$, since we choose any subset of the remaining $k/2$ steps to be LU steps.

Question 3. Determine the number of positive integers in [1000] which are divisible by 2 or by 3 or by 11.

Solution This is inclusion-exclusion. If A_i is the number of positive integers in [1000] divisible by a prime i, then we get that the answer is

$$|A_2 \cup A_3 \cup A_{11}| = |A_2| + |A_3| + |A_{11}| - |A_2 \cap A_3| - |A_2 \cap A_{11}| - |A_3 \cap A_{11}| + |A_2 \cap A_3 \cap A_{11}|.$$

Now $|A_i| = \lfloor 1000/i \rfloor$ for any i so we get $|A_2| = 500$, $|A_3| = 333$, $|A_{11}| = 90$. Now also $|A_i \cap A_j| = \lfloor 1000/ij \rfloor|$ when i and j are different primes, so $|A_2 \cap A_3| = 166$, $|A_2 \cap A_{11}| = 45$ and $|A_3 \cap A_{11}| = 30$. Finally, $|A_2 \cap A_3 \cap A_{11}| = 15$ so we get the answer is 697.

Question 4. Prove the inclusion-exclusion formula (Principle 3) by induction on n.

Solution. The formula is true for $n = 1$ and $n = 2$. Suppose it is true for $n = m$. Then

$$\Big|\bigcup_{i=1}^{m+1} A_i\Big| = \Big|A_{m+1} \cup \bigcup_{i=1}^{m} A_i\Big| = |A_{m+1}| + \Big|\bigcup_{i=1}^{m} A_i\Big| - \Big|A_{m+1} \cap \bigcup_{i=1}^{m} A_i\Big|.$$

The real issue here is the last term. By distributivity, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ we get that the last term is

$$\Big| A_{m+1} \cap \bigcup_{i=1}^{m} A_i \Big| = \Big| \bigcup_{i=1}^{m} (A_i \cap A_{m+1}) \Big|.$$

By induction, this is

$$\sum_{\emptyset \neq S \subseteq [m]} (-1)^{|S|} \Big| \bigcap_{i \in S} (A_i \cap A_{m+1}) \Big| = \sum_{\emptyset \neq S \subseteq [m]} (-1)^{|S|} \Big| \bigcap_{i \in S \cup [m+1]} A_i \Big|.$$

Also by induction,

$$\Big| \bigcup_{i=1}^{m} A_i \Big| = \sum_{\emptyset \neq S \subseteq [m]} (-1)^{|S|} \Big| \bigcap_{i \in S} A_i \Big|.$$

So finally putting the last two equations together we get

$$\Big| \bigcup_{i=1}^{m+1} A_i \Big| = |A_{m+1}| + \sum_{\emptyset \neq S \subseteq [m]} (-1)^{|S|} \Big| \bigcap_{i \in S} A_i \Big| + \sum_{\emptyset \neq S \subseteq [m]} (-1)^{|S|} \Big| \bigcap_{i \in S \cup [m+1]} A_i \Big|.$$

The only sets in $[m+1]$ not counted in the first sum are the sets containing $m+1$. But these sets are counted in the second sum, so we really have a single sum over all subsets of $[m+1]$, and we get

$$\Big| \bigcup_{i=1}^{m+1} A_i \Big| = \sum_{\emptyset \neq S \subseteq [m+1]} (-1)^{|S|} \Big| \bigcap_{i \in S} A_i \Big|.$$

This proves inclusion-exclusion.

Question 5. Let $\mathbb{R}$ denote the set of real numbers, and $(0,1)$ the set of real numbers strictly between zero and one. Find a bijection $g : (0,1) \to \mathbb{R}$.

Solution Take $g(x) = \tan((x - \frac{1}{2})\pi)$. Then $\lim g(x) = -\infty$ as $x \to 0^+$ and $\lim g(x) = \infty$ as $x \to 1^-$, and since g is continuous, g is onto $\mathbb{R}$. Clearly g is one-to-one since it is strictly increasing on $(0,1)$.

Question 6. Let A be the set of all sequences of positive integers of length k which add up to n, and let B be the set of all subsets of $[n-1]$ of size k. Find a bijection $f : A \to B$. Deduce that $|A| = \binom{n-1}{k-1}$ for $n \geq k \geq 1$.

Solution. For a sequence $(x_1, x_2, \dots, x_k) \in A$, define

$$f(x_1, x_2, \dots, x_k) = \{x_1, x_1 + x_2, \dots, x_1 + x_2 + \ldots + x_{k-1}\}.$$

Then since $x_1 + x_2 + \ldots + x_k = n$ and $x_k \geq 1$, we get $x_1 + x_2 + \ldots + x_{k-1} \leq n-1$ and so f has range in $[n-1]$. If $f(x_1, x_2, \dots, x_k) = f(y_1, y_2, \dots, y_k)$, then

$$x_1 = y_1 \quad x_1 + x_2 = y_1 + y_2 \quad \cdots \quad x_1 + x_2 + \ldots + x_{k-1} = y_1 + y_2 + \ldots + y_{k-1}$$

from which we get $x_i = y_i$ for $i < k$. But $x_k = n - x_1 - x_2 - \ldots - x_{k-1} = y_k$ in that case, and so we get $(x_1, x_2, \dots, x_k) = (y_1, y_2, \dots, y_k)$. In other words, f is one-to-one. To see that f is onto, if we are given a set $\{y_1, y_2, \ldots, y_{k-1}\} \subseteq [n-1]$, where $y_1 < y_2 < \ldots < y_{k-1}$, then

$$f(y_1, y_2 - y_1, y_3 - y_2, \ldots, y_{k-1} - y_{k-2}, n - y_{k-1}) = \{y_1, y_2, \ldots, y_k\}$$

because $y_1 + (y_2 - y_1) + (y_3 - y_2) + \ldots + (n - y_{k-1}) = n$. So f is onto and that means the number of sequences in $|A|$ is the number of sets of size $k-1$ in $[n-1]$, which is $\binom{n-1}{k-1}$.

Question 7. Let A be the set of all sequences of positive integers (of any length) which add up to n, and let B be the set of all subsets of $[n-1]$. Find a bijection $f : A \to B$. Deduce that $|A| = 2^{n-1}$ for $n \geq 1$.

Solution Let f be as in the last question, and then note $|A| = |B| = 2^{n-1}$.

Question 10. Prove the following identity by induction on n:

$$\sum_{k=1}^{n} k^3 = \binom{n+1}{2}^2.$$

Solution For $n = 1$ we get $1^3 = \binom{1+1}{2}^2$ which is valid. Suppose the statement holds for n, and now we prove

$$\sum_{k=1}^{n+1} k^3 = \binom{n+2}{2}^2.$$

First note

$$\sum_{k=1}^{n+1} k^3 = \sum_{k=1}^{n} k^3 + (n+1)^3 = \binom{n+1}{2}^2 + (n+1)^3$$

by induction. Now

$$\binom{n+1}{2}^2 + (n+1)^3 = \frac{(n+1)^2 n^2}{4} + (n+1)^3 = (n+1)^2\left(\frac{n^2}{4} + n + 1\right) = (n+1)^2\frac{(n+2)^2}{4} = \binom{n+2}{2}^2$$

as required.

Question 12. Prove, by induction on k, that the number of sequences of 1s and 2s of length k which add up to n is

$$\binom{k}{n-k}.$$

Solution This is true for $k = 1$ since in that case $n = 1$ or $n = 2$ and there is only $1 = \binom{1}{n-1}$ sequence. Passing to the induction step, consider a sequence of length $k+1$ of 1s and 2s. If we remove the last entry, we get a sequence of length $k-1$ which either adds up to $n-1$ or to $n-2$. By induction, there are $\binom{k-1}{n-k}$ sequences of length $k-1$ adding up to $n-1$ and there are $\binom{k-1}{n-k-1}$ sequences of length $k-1$ adding up to $n-2$. So what we need is

$$\binom{k}{n-k} = \binom{k-1}{n-k} + \binom{k-1}{n-k-1}$$

and this is true from Pascal's Triangle. Note we can find a bijection from sequences we want to the $n-k$ element subsets of $[k]$. Define

$$f(x_1, x_2, \dots, x_k) = \{i : x_i = 2\}.$$

Clearly f is a bijection since knowing $\{i : x_i = 2\}$ gives the whole sequence $(x_1, x_2, \dots, x_k)$. Furthermore $\{i : x_i = 2\}$ is a set of size $n-k$ and is contained in a set of size k.

Question 13. Let X be a set of size kn, where k and n are positive integers. An unordered partition of X into k-sets is a set $\{X_1, X_2, \dots, X_n\}$ such that $X_i \subseteq X$ is a set of size k, the sets X_i are disjoint, and the union of the X_i is X. Determine the number of unordered partitions of X into k-sets. For example, if $k = n = 2$, then he unordered partitions are

$$\{12, 34\} \quad \{13, 24\} \quad \{14, 23\}$$

where ij denotes the set $\{i, j\}$.

Solution There are $\binom{kn}{k}$ choices of X_1. Since X_2 is disjoint from X_1, there are $\binom{kn-k}{k}$ choices of X_2. In general, there are $\binom{kn-ik}{k}$ choices of X_{i+1}, because when X_{i+1} is chosen, we have to remove $X_1, X_2, \dots, X_i$ which cover ik elements out of $[kn]$. So the number of **sequences** $(X_1, X_2, \dots, X_n)$ is

$$\binom{kn}{k}\binom{kn-k}{k}\cdots\binom{k}{k}.$$

To get the number of **sets** $\{X_1, X_2, \dots, X_n\}$, we note that there are $n!$ sequences from each set, so the sequences overcount the sets by $n!$. Therefore the number of sets is

$$\frac{1}{n!}\binom{kn}{k}\binom{kn-k}{k}\cdots\binom{k}{k}.$$

This can be simplified a bit to

$$\frac{(kn)!}{k!^n n!}.$$

Question 14* Let X be an n-element set, where $n \geq 2$. Suppose we choose some subsets of X such that no two chosen subsets share two points and each chosen subset has at least two elements. Prove that there are distinct elements $x, y \in X$ which are in the same number of chosen subsets. For example, in the Figure 5, we have $|X| = n = 15$ (dots denote elements of X), ten chosen subsets (seven of size two, one of size three, one of size four, one of size five), and we notice that the leftmost and rightmost dots are each in three of the chosen sets.

Solution For $x \in X$ let $d(x)$ be the number of chosen subsets containing x. We want $d(x) = d(y)$ for two points $x, y \in X$. If $d(x) \geq n$ for some x, then since there are only $n-1$ points other than x and $n \geq 2$, we would get two chosen subsets both containing x and some other point, which is not allowed. So $d(x) < n$ for every x. If $d(x) \geq 1$ for all x, then $d(x) \in [n-1]$ for all x but there are n possible x. So the pigeonhole principle says $d(x) = d(y)$ for some two points x, y. If $d(x) = 0$ for some x, then $d(y) \leq n-2$ for every $y \in X$, otherwise some chosen set would contain x if $d(y) = n - 1$ for some y. So $0 \leq d(y) \leq n - 2$ for all $y \in X$, and again the pigeonhole principle gives the desired result.

As a special case, this result says that every graph with $n \geq 2$ vertices contains two vertices of the same degree.

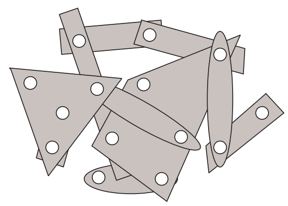

Figure 5 : Chosen subsets in a fifteen-element set.