
Question 1. Determine each of the following quantities explicitly.

(a) $\binom{-\frac{1}{2}}{4}$ (b) $\binom{-\frac{1}{3}}{3}$ (c) $\sum_{k=1}^{\infty}\binom{\frac{1}{3}}{k}\left(\frac{1}{7}\right)^k$ (d) $\sum_{k=1}^{\infty}\frac{1}{k2^k}$

Solution 1.

(a) $\binom{-\frac{1}{2}}{4} = \frac{(-\frac{1}{2})(-\frac{3}{2})(-\frac{5}{2})(-\frac{7}{2})}{24} = \frac{105}{384}$.

(b) $\binom{-\frac{1}{3}}{3} = \frac{(-\frac{1}{3})(-\frac{4}{3})(-\frac{7}{3})}{6} = \frac{-28}{162} = \frac{-14}{81}$.

(c) By the binomial theorem

$$\sum_{k=0}^{\infty}\binom{\frac{1}{3}}{k}7^{-k} = \left(\frac{8}{7}\right)^{\frac{1}{3}}$$

so we subtract 1 from this to get the answer $(\frac{8}{7})^{\frac{1}{3}} - 1$.

(d) The key is formal integration:

$$\begin{aligned}\int \frac{1}{1-x}dx &= \int \sum_{k=1}^{\infty} x^{k-1}dx \\ &= \sum_{k=1}^{\infty}\int x^{k-1}dx \\ &= \sum_{k=1}^{\infty}\frac{x^k}{k}.\end{aligned}$$

The first integral is formally $-\ln(1-x)$, so if we put $x = \frac{1}{2}$, we get

$$\ln 2 = -\ln(1-\frac{1}{2}) = \sum_{k=1}^{\infty}\frac{1}{k2^k}.$$

Question 2. Prove that the inverse of $1 + 2x + 3x^2 + 2x^3 + x^4$ is

$$\sum_{j=1}^{\infty}\sum_{i=1}^{j}(-1)^{j-1}i\binom{j}{i}x^{i+j-2}.$$

Solution 2. One can either multiply these two formal power series together and show that everything cancels except the constant term of 1, or observe $(1+x+x^2)^2 = 1+2x+3x^2+2x^3+x^4$. We use this route. By the binomial theorem

$$\begin{aligned}(1+x+x^2)^{-2} &= \sum_{j=0}^{\infty}\binom{-2}{j}(x+x^2)^j \\ &= \sum_{j=0}^{\infty}(j+1)(-1)^j x^j(1+x)^j \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{j}(j+1)\binom{j}{i}(-1)^j x^{i+j} \\ &= \sum_{j=0}^{\infty}\sum_{i=0}^{j}(i+1)\binom{j+1}{i+1}(-1)^j x^{i+j} \\ &= \sum_{j=1}^{\infty}\sum_{i=1}^{j} i\binom{j}{i}(-1)^{j-1} x^{i+j-2}\end{aligned}$$

where we used $\binom{-2}{j} = (-1)^j(j+1)$ and $\binom{j+1}{i+1} = \frac{j+1}{i+1}\binom{j}{i}$.

Question 3. The range of a non-empty subset of $[n]$ is the difference between the largest and smallest elements of of the set.

(a) Prove that the number of subsets of $[n]$ whose range is k is exactly $(n-k)2^{k-1}$ for $k \geq 1$ and n for $k = 0$.

(b) If S is the set of non-empty subsets of $[n]$ and the weight of a set is its range, determine the generating function $\Phi_S(x)$.

(c) Determine the average range of subsets of $[n]$.

Solution 3.

(a) For $k = 0$ the sets with range k are the singletons $\{i\} : i \in [n]$, and there are n of them. There are $n-k$ ways to choose the smallest and largest element of a set whose range is $k \geq 1$: the possibilities are $i, i+k$ where $i \leq n-k$. Having chosen the smallest and largest element, we can add any subset of the $k-1$ elements inbetween, and there are 2^{k-1} such subsets. Therefore $(n-k)2^{k-1}$ sets have range k in $[n]$.

(b) From (a), the generating function is

$$\Phi(x) = n + \sum_{k=1}^{n}(n-k)2^{k-1}x^k.$$

(c) By a theorem in class, the average range is $\Phi'(1)/\Phi(1)$ and we already know $\Phi(1) = 2^n - 1$. Define

$$\begin{aligned} f_1(x) &= \frac{d}{dx}\frac{1-x^{n+1}}{1-x} = \frac{nx^{n+1}-(n+1)x^n+1}{(1-x)^2} \\ f_2(x) &= \frac{d}{dx}f_1(x) = \frac{(n-n^2)x^{n+1}+(2n^2-2)x^n-(n^2+n)x^{n-1}+2}{(1-x)^3}. \end{aligned}$$

Then from part (b),

$$\begin{aligned} \Phi'(1) &= \sum_{k=1}^{n} k(n-k)2^{k-1} \\ &= n\sum_{k=1}^{n} k2^{k-1} - \sum_{k=1}^{n} k^2 2^{k-1} \\ &= n\sum_{k=1}^{n} k2^{k-1} - \sum_{k=1}^{n} k(k-1)2^{k-1} - \sum_{k=1}^{n} k2^{k-1} \\ &= (n-1)\sum_{k=1}^{n} k2^{k-1} - 2\sum_{k=2}^{n} k(k-1)2^{k-2} \\ &= (n-1)f_1(2) - 2f_2(2). \end{aligned}$$

Here we used the formal derivative of a power series, in the sense that

$$f_1(x) = \sum_{k=1}^{n} kx^{k-1} \quad \text{and} \quad f_2(x) = \sum_{k=2}^{n} k(k-1)x^{k-2}.$$

So finally, computing $f_1(2)$ and $f_2(2)$, we get

$$\frac{\Phi'(1)}{\Phi(1)} = \frac{(n-3)2^n+n+3}{2^n-1}$$

and the average range is about $n-3$ when n is large.

Question 4. Determine the number of compositions of n into k parts such that the ith part is i or $i+1$.

Solution 4. We are counting sequences in the set

$$S = \{(x_1, x_2, \ldots, x_k) : x_i \in \{i, i+1\} \text{ for } i \in [k]\}.$$

Thus $S = S_1 \times S_2 \times \ldots \times S_k$ where $S_i = \{i, i+1\}$ for $i \in [k]$. By definition the generating function for S_i is

$$\Phi_{S_i}(x) = x^i + x^{i+1} \quad \text{for } i \in [k]$$

and according to the product lemma,

$$\Phi_S(x) = \prod_{i=1}^{k} \Phi_{S_i}(x) = \prod_{i=1}^{k}(x^i + x^{i+1}) = (1+x)^k x^{\sum_{i=1}^{k} i}.$$

In the exponent we have the sum of the first k positive integers, which is $\binom{k+1}{2}$, so

$$\Phi_S(x) = (1+x)^k x^{\binom{k+1}{2}}.$$

By the binomial theorem,

$$\Phi_S(x) = \sum_{j=0}^{k} \binom{k}{j} x^{j+\binom{k+1}{2}}.$$

Therefore the coefficient of x^n in $\Phi_S(x)$ is

$$[x^n]\Phi_S(x) = \binom{k}{n-\binom{k+1}{2}}$$

provided that $n \geq \binom{k+1}{2}$. If $n < \binom{k+1}{2}$, then the answer is zero. So we conclude that if a_n is the number of compositions as described, then

$$a_n = \begin{cases} \binom{k}{n-\binom{k+1}{2}} & \text{if } n \geq \binom{k+1}{2} \\ 0 & \text{otherwise} \end{cases}$$