
Question 1. Determine the exact numerical values of the following binomial coefficients:

(a) $\binom{1}{2}$
(b) $\binom{2}{0}$
(c) $\binom{n}{n-1}$
(d) $\binom{1/2}{4}$
(e) $\binom{-1/2}{4}$
(f) $\binom{-1/3}{3}$
(g) $\binom{1/3}{k}$.

Solutions.
(a) 0 (b) 1 (c) n (d) $-\frac{5}{128}$ (e) $\frac{35}{128}$ (f) $-\frac{14}{81}$ (g) $\frac{1\cdot(-2)\cdot(-5)\cdots(4-3k)}{k!3^k}$ for $k > 0$ and 1 for $k = 0$.

Question 2. Find the inverses of the following formal power series as a sum of powers of x, or state that the power series has no inverse. Justify your answers.

(a) $1 - 2x + x^2$
(b) $x(1 - x)$
(c) $4 - x^2$
(d) $1 - x + x^2 - x^3 + x^4 - x^5 + \dots$
(e) $1 + 2x + 3x^2 + 4x^3 + 5x^4 + \dots$
(f) $1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$.

Solutions.
(a) $1 - 2x + x^2 = (1 - x)^2$. Therefore the inverse is

$$(1-x)^{-2} = \sum_{j=0}^{\infty} \binom{-2}{j} (-x)^j = \sum_{j=0}^{\infty} (j+1)x^j$$

from the binomial theorem.

(b) This has no inverse since it has a zero constant term (see theorem in the notes).

(c) $(4 - x^2)^{-1} = \frac{1}{4}(1 - \frac{x^2}{4})^{-1} = \frac{1}{4}\sum_{j=0}^{\infty}(\frac{x^2}{4})^j$ (geometric series).

(d) $1 - x + x^2 - x^3 + \dots = \sum_{j=0}^{\infty}(-1)^j x^j = \frac{1}{1+x}$. Therefore $1 + x$ is the inverse.

(e) $1 + 2x + 3x^2 + 4x^3 + \dots = \sum_{j=0}^{\infty}(j + 1)x^j$. Now we know from part (a) that the inverse is $1 - 2x + x^2$.

(f) This is $A(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$. We know from calculus that this is e^x if x has a real value. So it makes sense to use the Taylor expansion of e^{-x} as an inverse: the inverse we guess is

$$B(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = 1 - x + \frac{x^2}{2!} - \cdots$$

and we can check this directly by multiplying the two formal power series. We see that

$$[x^n]A(x)B(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!}\frac{1}{(n-k!)} = \frac{1}{n!}\sum_{k=0}^{n}(-1)^k\binom{n}{k} = \frac{1}{n!}(1-1)^n = 0$$

provided $n \neq 0$. For $n = 0$ we get 1. So we conclude $A(x)B(x)$ has all terms zero except $1 \cdot x^0 = 1$, and so $B(x)$ is the inverse of $A(x)$.

Question 3. Determine the generating function in closed form for the given set S of configurations with weight function ω.

(a) S is the set of subsets of $[n]$, $\omega(\sigma) = |\sigma|$
(b) $S = \mathbb{Z}^+$ and $\omega(\sigma) = 2\sigma$.
(c) $S = \mathbb{Z}^+$ and $\omega(\sigma) = \sigma$ if σ is odd and $\omega(\sigma) = 0$ otherwise.
(d) $S = [n]$ and $\omega(\sigma) = 1$ if σ is odd and $\omega(\sigma) = 0$ if σ is even.
(e) S is the set of permutations of $[4]$, $\omega(\sigma)$ is the number of fixed points of σ.
(f) S is the set of pairs (a, b) of positive integers, and $\omega(a, b) = a + b$.
(g) S is the set of pairs (a, b) of positive integers, and $\omega(a, b) = 2a + b$.

Solutions.
(a) We did this in class, it is $(1 + x)^n$ since $a_k = \binom{n}{k}$ is the number of $\sigma \in S$ with $\omega(\sigma) = k$.

(b) We get $\Phi_S(x) = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2}$.

(c) Since $a_0 = \infty$ (there are infinitely many things of weight zero, namely all even numbers) the generating function does not exist.

(d) In this case $a_0 = \lfloor \frac{n}{2} \rfloor$ and $a_1 = \lceil \frac{n}{2} \rceil$ and $a_k = 0$ otherwise. Therefore

$$\Phi(x) = \lfloor \frac{n}{2} \rfloor + \lceil \frac{n}{2} \rceil x.$$

(e) From the notes, there are $\lfloor n!/e + 1/2 \rfloor$ derangements of n elements. Now that means $\lfloor 4!/e + 1/2 \rfloor = 9$ permutations have zero fixed points (weight zero). The number of permutations with one fixed point is $4 \cdot \lfloor 3!/e + 1/2 \rfloor = 8$ since there are 4 choices for where the fixed point is and $\lfloor 3!/e + 1/2 \rfloor$ ways to fill in the rest with no more fixed points. Similarly there are $\binom{4}{2}\lfloor 2!/e + 1/2 \rfloor = 6$ permutations with two fixed points. There are no permutations with exactly three fixed points and one permutation with four fixed points. Therefore

$$\Phi(x) = 9 + 8x + 6x^2 + x^4.$$

(f) $\Phi(x) = \sum_{(a,b)} x^{a+b} = \sum_{a=1}^{\infty} x^a \sum_{b=1}^{\infty} x^b = \frac{x^2}{(1-x)^2}$.

(g) $\Phi(x) = \sum_{(a,b)} x^{2a+b} = \sum_{a=1}^{\infty} x^{2a} \sum_{b=1}^{\infty} x^b = \frac{x^3}{(1-x)(1-x^2)}$.

Question 4. Compute the average maximum element of non-empty subsets of $[n]$.

Solution. Let S be the set of nonempty subsets of n and $\omega(\sigma) = \max \sigma$. There are 2^{k-1} sets in S of weight k – having put k, the maximum element, into the set, we can choose any subset of $[k-1]$ to add to $\{k\}$ to get a set with maximum element equal to k. Therefore

$$\Phi_S(x) = \sum_{k=1}^{n} 2^{k-1}x^k = \frac{x(1-(2x)^n)}{1-2x}.$$

The average maximum element is $\Phi'_S(1)/\Phi_S(1)$ according to a theorem in the notes. Clearly $\Phi_S(1) = 2^n - 1$. Now

$$\Phi'_S(x) = \frac{n(2x)^{n+1} - (n+1)(2x)^n + 1}{(1-2x)^2}.$$

Therefore

$$\Phi'_S(1) = n2^{n+1} - (n+1)2^n + 1$$

and the average maximum is

$$\frac{n2^{n+1} - (n+1)2^n + 1}{2^n - 1}.$$

This is very close to $n - 1$.

Question 5. What is the average difference between the largest and smallest elements of a non-empty subset of $[n]$?

Solution. This was on the last assignment (average range).

Question 6. Determine the number of compositions of n into k parts with the given restrictions.

(a) Each part is a positive even integer.
(b) Each part is an element of $\{2, 3\}$
(c) Each part is at most i
(d) Exactly one part is odd, the rest are positive and even.

Solutions.
(a) $S = S_1 \times S_2 \times \cdots \times S_k$ where $S_i = \{2, 4, 6, \ldots\}$. Now

$$\Phi_{S_i}(x) = x^2 + x^4 + x^6 + \ldots = \frac{x^2}{1-x^2}.$$

Therefore by the product lemma

$$\Phi_S(x) = x^{2k}(1-x^2)^{-k}.$$

The binomial theorem gives

$$\Phi_S(x) = x^{2k} \sum_{j=0}^{\infty} \binom{-k}{j} (-1)^j x^{2j}.$$

Therefore the number of compositions we want is

$$[x^n]\Phi_S(x) = \binom{-k}{n/2-k}(-1)^{n/2-k}$$

provided n is even, and zero otherwise. Here we used $2j + 2k = n$ in the sum.

(b) $S = \{2,3\} \times \{2,3\} \times \cdots \times \{2,3\}$ a total of k times. Since $\Phi_{\{2,3\}}(x) = x^2 + x^3$ we get

$$\Phi_S(x) = (x^2 + x^3)^k$$

from the product lemma. Then the binomial theorem gives

$$\Phi_S(x) = x^{2k} \sum_{j=0}^{k} \binom{k}{j} x^j.$$

Finally $[x^n]\Phi_S(x) = \binom{k}{n-2k}$ which is valid for $2k \leq n$. If $2k > n$ then the answer is zero.

(c) $S = [i] \times [i] \times \cdots \times [i]$. Now $\Phi_{[i]}(x) = x + x^2 + \ldots + x^i$ which is $x(1-x^i)/(1-x)$ by finite geometric series. So

$$\Phi_S(x) = x^k(1-x^i)^k(1-x)^{-k}.$$

By the binomial theorem

$$\Phi_S(x) = x^k \sum_{r=0}^{k} \sum_{s=0}^{\infty} \binom{k}{r}(-x^i)^r \binom{-k}{s}(-x)^s.$$

To get $[x^n]\Phi_S(x)$ take $ir + s + k = n$ i.e. take $s = n - k - ir$. Then the answer is

$$\sum_{r=0}^{k} \binom{k}{r}(-1)^r \binom{-k}{n-k-ir}(-1)^{n-k-ir}.$$

Question 7. Determine the number of compositions of n into any number of parts, where each part is odd.

Solution. Let $S(k)$ be the set of sequences of positive integers of length k and $S = \bigcup_{k=0}^{\infty} S(k)$. Then

$$S(k) = S_1 \times S_2 \times \cdots \times S_k$$

where $S_i = \{1, 3, 5, \ldots\}$ for all i. Now

$$\Phi_{S_i}(x) = x + x^3 + x^5 + \ldots = \frac{x}{1-x^2}$$

since it is an infinite geometric series. By the product lemma

$$\Phi_{S(k)}(x) = \left(\frac{x}{1-x^2}\right)^k.$$

By the sum lemma

$$\Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{S(k)}(x) = \sum_{k=0}^{\infty} \left(\frac{x}{1-x^2} \right)^k = \frac{1}{1 - \frac{x}{1-x^2}}$$

since it is an infinite geometric series. Simplifying, we get

$$\Phi_S(x) = \frac{1-x^2}{1-x-x^2}.$$

We can proceed by using recurrence equations or sums. By recurrence equations we get

$$a_n = a_{n-1} + a_{n-2}$$

where $a_1 = 1$ and $a_2 = 1$. Solving this (show all working as in the notes) we get the Fibonacci numbers

$$a_n = \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left(\frac{1-\sqrt{5}}{2} \right)^{n+1}.$$

Doing it by sums, we get

$$\Phi_S(x) = (1-x^2) \sum_{j=0}^{\infty} (x+x^2)^j$$

since the sum of the geometric series is $1/(1-x-x^2)$, and now

$$\Phi_S(x) = (1-x^2) \sum_{j=0}^{\infty} \sum_{i=0}^{j} x^j \binom{j}{i} x^i.$$

To get $[x^n]\Phi_S(x)$ split the sum in two:

$$\Phi_S(x) = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i} x^{i+j} - \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i} x^{i+j+2}.$$

In the first let $i = n-j$ and in the second let $i = n-j-2$ to get

$$[x^n]\Phi_S(x) = \sum_{j=0}^{n} \binom{j}{n-j} - \sum_{j=0}^{n-2} \binom{j}{n-j-2}.$$

Question 8. How many sets $\{x_1, x_2, \ldots, x_k\} \subset [n]$ have $x_{i-1} + i \leq x_i$ for $2 \leq i \leq k$?

Solution. If we consider the sequence $(y_1, y_2, \ldots, y_{k+1}) = (x_1, x_2 - x_1, \ldots, x_k - x_{k-1}, n - x_k)$ then we have a composition of n with $k+1$ part and where the ith part is at least i except that the last part is any non-negative integer (since $n - x_k \geq 0$). Let

$$S = S_1 \times S_2 \times \ldots \times S_{k+1}$$

where $S_{k+1} = \mathbb{Z}^+$ and $S_i = \{i, i+1, i+2, \ldots\}$ for $i \in [k]$. Then for $i \in [k]$,

$$\Phi_{S_i}(x) = x^i + x^{i+1} + \ldots = \frac{x^i}{1-x}$$

and

$$\Phi_{S_{k+1}}(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}.$$

Therefore by the product lemma

$$\Phi_S(x) = \prod_{i=1}^{k} \frac{x^i}{1-x} \cdot \frac{1}{1-x} = x^{\binom{k+1}{2}}(1-x)^{-k-1}.$$

Here we used $x^1 \cdot x^2 \cdots x^k = x^{1+2+\ldots+k}$ and $1+2+\ldots+k = \binom{k+1}{2}$. Now the binomial theorem gives

$$\Phi_S(x) = x^{\binom{k+1}{2}} \sum_{j=0}^{\infty} \binom{-k-1}{j} (-x)^j.$$

To get $[x^n]\Phi_S(x)$ put $j = n - \binom{k+1}{2}$ so that

$$[x^n]\Phi_S(x) = \binom{-k-1}{n - \binom{k+1}{2}} (-1)^{n-\binom{k+1}{2}}.$$

Question 9* Let S denote the set of unordered lists (sets with repeated elements allowed) of positive integers, with weight function $\omega(\{x_1, x_2, \ldots, x_k\}) = x_1 + x_2 + \ldots + x_k$. Prove that

$$\Phi_S(x) = \prod_{r=1}^{\infty} \frac{1}{1-x^r}.$$

Solution. Since

$$\frac{1}{1-x^r} = 1 + x^r + x^{2r} + \ldots$$

we get that

$$\Phi_S(x) = (1+x+x^2+\cdots)(1+x^2+x^4+\cdots)(1+x^3+x^6+\cdots)(1+x^4+x^8+\cdots)\cdots.$$

Then $[x^n]\Phi_S(x)$ should be the number of sets of integers whose sum of elements is n. Now each a term from the rth bracket has the form x^{ir} so

$$n = m_1 + 2m_2 + 3m_3 + \ldots + rm_r.$$

This means we are writing n as a sum of elements in the list $1111112222223333 3\cdots rrrrr$ where i appears m_i times as required.

Question 10. Determine the number of binary strings of length n with the given restrictions.

(a) The strings have only blocks of odd length.
(b) The strings do not contain 011.
(c) The strings do not contain 101 or 010.
(d)* The strings do not contain 101.

Solutions.
(a) The expression

$$S = (\{\varepsilon\} \cup \{0\}\{00\}^*)(\{1\}\{11\}^*\{0\}\{00\}^*)^*(\{\varepsilon\} \cup \{1\}\{11\}^*)$$

uniquely creates all the strings we want. The generating function is

$$\Phi_S(x) = \left(1 + \frac{x}{1-x^2}\right)^2 \frac{1}{1 - \left(\frac{x}{1-x^2}\right)^2} = \frac{(1+x-x^2)^2}{1-3x^2+x^4}.$$

By the binomial theorem

$$\Phi_S(x) = (1+x-x^2)^2 \sum_{j=0}^{\infty}(3x^2-x^4)^j = (1+x-x^2)^2 \sum_{j=0}^{\infty}\sum_{i=0}^{j}(3x^2)^j \binom{j}{i}\left(-\frac{x^2}{3}\right)^i.$$

It is particularly unpleasant to find $a_n = [x^n]\Phi_S(x)$ from this but it can be done. The other way is to use the recurrence

$$a_n - 3a_{n-2} + a_{n-4} = 0.$$

Note that $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 4$ (since $111, 000, 101, 010$ are the strings of length three with all blocks of odd length). The characteristic equation is $x^4 - 3x^2 + 1 = 0$. The roots are

$$\sqrt{\frac{3 \pm \sqrt{5}}{2}} \quad \text{and} \quad -\sqrt{\frac{3 \pm \sqrt{5}}{2}}$$

which means

$$a_n = c_1\sqrt{\frac{3 \pm \sqrt{5}}{2}}^{\,n} - c_2\sqrt{\frac{3+\sqrt{5}}{2}}^{\,n} + c_3\sqrt{\frac{3-\sqrt{5}}{2}}^{\,n} - c_4\sqrt{\frac{3-\sqrt{5}}{2}}^{\,n}.$$

Here c_1, c_2, c_3, c_4 are constants to be determined using a_0, a_1, a_2, a_3.

s (b) Generating function is

$$\frac{1}{1-2x+x^3} = \sum_{j=0}^{\infty}(2x-x^3)^j = \sum_{j=0}^{\infty}\sum_{i=0}^{j}(2x)^j(-x^2/2)^i\binom{j}{i}.$$

The coefficient of x^n is

$$\sum_{i=0}^{n}\binom{n-2i}{i}(-1)^i 2^{n-3i}.$$

Alternative: use recurrence equation $a_n = 2a_{n-1} - a_{n-3}$ with initial conditions $a_0 = 1$, $a_1 = 2$ and $a_2 = 4$. The characteristic equation is $\alpha^3 - 2\alpha^2 + 1 = 0$ and $\alpha = 1$ is an obvious root. Then we get $(\alpha - 1)(\alpha^2 - \alpha - 1) = 0$ so $\alpha = \frac{1}{2}(1 \pm \sqrt{5})$. Then

$$a_n = c_1 + c_2 \left(\frac{1+\sqrt{5}}{2}\right)^n + c_3 \left(\frac{1-\sqrt{4}}{2}\right)^n.$$

Finally c_1, c_2, c_3 are determined from $a_0 = 1, a_1 = 2$ and $a_2 = 4$. Thus

$$c_1+c_2+c_3 = 1 \quad c_1+c_2\left(\frac{1+\sqrt{5}}{2}\right)+c_3\left(\frac{1-\sqrt{5}}{2}\right) = 2 \quad c_1+c_2\left(\frac{1+\sqrt{5}}{2}\right)^2+c_3\left(\frac{1-\sqrt{5}}{2}\right)^2 = 4.$$

Solving these equations (e.g. by linear algebra) we get $c_1 = -1$ and $c_2 = (5+2\sqrt{5})/5$ and $c_3 = (5-2\sqrt{5})/5$. Therefore

$$a_n = 1 + \frac{5+2\sqrt{5}}{5}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{5-2\sqrt{5}}{5}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

(c) These strings are uniquely created by

$$S = \{0\}^*(\{11\}\{1\}^*\{00\}\{0\}^*)^*\{1\}^* \cup \{1\}\{1\}^*(\{00\}\{0\}^*\{11\}\{1\}^*)^*\{0\}^*.$$

The generating function is

$$\begin{aligned}\Phi_S(x) &= \left(\frac{1}{(1-x)^2} + x\right) \cdot \left(\frac{1}{1 - \frac{x^4}{(1-x)^2}}\right)\\ &= \frac{1}{1-2x+x^2-x^4} + \frac{x(1-x)^2}{1-2x+x^2-x^4}\\ &= \frac{1+x-2x^2+x^3}{1-2x+x^2-x^4}.\end{aligned}$$

Opting for recurrences, we get $a_n - 2a_{n-1} + a_{n-2} - a_{n-4} = 0$ and $a_0 = 1$, $a_1 = 2$, $a_2 = 4$, $a_3 = 6$. The characteristic equation has four distinct roots: $(1 \pm \sqrt{5})/2$ and $(1 \pm \sqrt{3}i)/2$. So

$$a_n = c_1\left(\frac{1+\sqrt{3}i}{2}\right)^n + c_2\left(\frac{1-\sqrt{3}i}{2}\right)^n + c_3\left(\frac{1+\sqrt{5}}{2}\right)^n + c_4\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Use the initial conditions now to find c_1, c_2, c_3, c_4.