**Question 1.** Determine the exact numerical values of the following binomial coefficients:

(a) \( \binom{1}{2} \)
(b) \( \binom{n}{0} \)
(c) \( \binom{n}{n-1} \)
(d) \( \binom{1/2}{4} \)
(e) \( \binom{-1/2}{4} \)
(f) \( \binom{-1/3}{3} \)
(g) \( \binom{1/3}{k} \).

**Solutions.**

(a) 0 (b) 1 (c) \( n \) (d) \( -\frac{5}{128} \) (e) \( \frac{35}{128} \) (f) \( -\frac{14}{81} \) (g) \( 1 \cdot (-2) \cdot (-5) \cdots (4-3k) \) for \( k > 0 \) and 1 for \( k = 0 \).

**Question 2.** Find the inverses of the following formal power series as a sum of powers of \( x \), or state that the power series has no inverse. Justify your answers.

(a) \( 1 - 2x + x^2 \)
(b) \( x(1 - x) \)
(c) \( 4 - x^2 \)
(d) \( 1 - x + x^2 - x^3 + x^4 - x^5 + \ldots \)
(e) \( 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \ldots \)
(f) \( 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots \)

**Solutions.**

(a) \( 1 - 2x + x^2 = (1 - x)^2 \). Therefore the inverse is

\[
(1 - x)^{-2} = \sum_{j=0}^{\infty} \binom{-2}{j} (-x)^j = \sum_{j=0}^{\infty} (j + 1) x^j
\]

from the binomial theorem.

(b) This has no inverse since it has a zero constant term (see theorem in the notes).

(c) \( (4 - x^2)^{-1} = \frac{1}{4} (1 - \frac{x^2}{4})^{-1} = \frac{1}{4} \sum_{j=0}^{\infty} \binom{4}{j} (-\frac{x^2}{4})^j \) (geometric series).

(d) \( 1 - x + x^2 - x^3 + \ldots = \sum_{j=0}^{\infty} (-1)^j x^j = \frac{1}{1+x} \). Therefore \( 1 + x \) is the inverse.

(e) \( 1 + 2x + 3x^2 + 4x^3 + \ldots = \sum_{j=0}^{\infty} (j + 1) x^j \). Now we know from part (a) that the inverse is \( 1 - 2x + x^2 \).

(f) This is \( A(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!} \). We know from calculus that this is \( e^x \) if \( x \) has a real value. So it makes sense to use the Taylor expansion of \( e^{-x} \) as an inverse: the inverse we guess is

\[
B(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^k}{k!} = 1 - x + \frac{x^2}{2!} - \ldots
\]
and we can check this directly by multiplying the two formal power series. We see that

\[ [x^n]A(x)B(x) = \sum_{k=0}^{n} \frac{(-1)^k}{k!} \frac{1}{(n-k)!} = \frac{1}{n!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} = \frac{1}{n!} (1-1)^n = 0 \]

provided \( n \neq 0 \). For \( n = 0 \) we get 1. So we conclude \( A(x)B(x) \) has all terms zero except \( 1 \cdot x^0 = 1 \), and so \( B(x) \) is the inverse of \( A(x) \).

**Question 3.** Determine the generating function in closed form for the given set \( S \) of configurations with weight function \( \omega \).

(a) \( S \) is the set of subsets of \([n] \), \( \omega(\sigma) = |\sigma| \)
(b) \( S = \mathbb{Z}^+ \) and \( \omega(\sigma) = 2\sigma \).
(c) \( S = [n] \) and \( \omega(\sigma) = \sigma \) if \( \sigma \) is odd and \( \omega(\sigma) = 0 \) otherwise.
(d) \( S = [n] \) and \( \omega(\sigma) = 1 \) if \( \sigma \) is odd and \( \omega(\sigma) = 0 \) if \( \sigma \) is even.
(e) \( S \) is the set of permutations of \([4] \), \( \omega(\sigma) \) is the number of fixed points of \( \sigma \).
(f) \( S \) is the set of pairs \((a, b)\) of positive integers, and \( \omega(a, b) = a + b \).
(g) \( S \) is the set of pairs \((a, b)\) of positive integers, and \( \omega(a, b) = 2a + b \).

**Solutions.**

(a) We did this in class, it is \((1 + x)^n\) since \( a_k = \binom{n}{k} \) is the number of \( \sigma \in S \) with \( \omega(\sigma) = k \).

(b) We get \( \Phi_S(x) = \sum_{k=0}^{\infty} x^{2k} = \frac{1}{1-x^2} \).

(c) Since \( a_0 = \infty \) (there are infinitely many things of weight zero, namely all even numbers) the generating function does not exist.

(d) In this case \( a_0 = \lceil \frac{n}{2} \rceil \) and \( a_1 = \lfloor \frac{n}{2} \rfloor \) and \( a_k = 0 \) otherwise. Therefore

\[ \Phi(x) = \lceil \frac{n}{2} \rceil + \lfloor \frac{n}{2} \rfloor x. \]

(e) From the notes, there are \( \lfloor n!/e + 1/2 \rfloor \) derangements of \( n \) elements. Now that means \( \lfloor 4!/e + 1/2 \rfloor = 9 \) permutations have zero fixed points (weight zero). The number of permutations with one fixed point is \( 4 \cdot \lfloor 3!/e + 1/2 \rfloor = 8 \) since there are 4 choices for where the fixed point is and \( \lfloor 3!/e + 1/2 \rfloor \) ways to fill in the rest with no more fixed points. Similarly there are \( \binom{4}{2} \lfloor 2!/e + 1/2 \rfloor = 6 \) permutations with two fixed points. There are no permutations with exactly three fixed points and one permutation with four fixed points. Therefore

\[ \Phi(x) = 9 + 8x + 6x^2 + x^4. \]

(f) \( \Phi(x) = \sum_{(a, b)} x^{a+b} = \sum_{a=1}^{\infty} x^a \sum_{b=1}^{\infty} x^b = \frac{x^2}{(1-x)^2} \)

(g) \( \Phi(x) = \sum_{(a, b)} x^{2a+b} = \sum_{a=1}^{\infty} x^{2a} \sum_{b=1}^{\infty} x^b = \frac{x^3}{(1-x)(1-x^2)} \).
**Question 4.** Compute the average maximum element of non-empty subsets of $[n]$.

**Solution.** Let $S$ be the set of nonempty subsets of $n$ and $\omega(\sigma) = \max \sigma$. There are $2^{k-1}$ sets in $S$ of weight $k$ – having put $k$, the maximum element, into the set, we can choose any subset of $[k-1]$ to add to $\{k\}$ to get a set with maximum element equal to $k$. Therefore

$$\Phi_S(x) = \sum_{k=1}^{n} 2^{k-1} x^k = \frac{x(1 - (2x)^n)}{1 - 2x}.$$  

The average maximum element is $\Phi'_S(1)/\Phi_S(1)$ according to a theorem in the notes. Clearly $\Phi_S(1) = 2^n - 1$. Now

$$\Phi'_S(x) = \frac{n(2x)^{n+1} - (n+1)(2x)^n + 1}{(1 - 2x)^2}.$$  

Therefore

$$\Phi'_S(1) = n2^{n+1} - (n+1)2^n + 1$$  

and the average maximum is

$$\frac{n2^{n+1} - (n+1)2^n + 1}{2^n - 1}.$$  

This is very close to $n - 1$.

**Question 5.** What is the average difference between the largest and smallest elements of a non-empty subset of $[n]$?

**Solution.** This was on the last assignment (average range).

**Question 6.** Determine the number of compositions of $n$ into $k$ parts with the given restrictions.

(a) Each part is a positive even integer.
(b) Each part is an element of $\{2, 3\}$
(c) Each part is at most $i$
(d) Exactly one part is odd, the rest are positive and even.

**Solutions.**

(a) $S = S_1 \times S_2 \times \cdots \times S_k$ where $S_i = \{2, 4, 6, \ldots\}$. Now

$$\Phi_{S_i}(x) = x^2 + x^4 + x^6 + \ldots = \frac{x^2}{1 - x^2}.$$  

Therefore by the product lemma

$$\Phi_S(x) = x^{2k}(1 - x^2)^{-k}.$$  

The binomial theorem gives

$$\Phi_S(x) = x^{2k} \sum_{j=0}^{\infty} \binom{-k}{j} (-1)^j x^{2j}.$$  

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Therefore the number of compositions we want is

\[ [x^n] \Phi_S(x) = \left( \frac{-k}{n/2-k} \right) (-1)^{n/2-k} \]

provided \( n \) is even, and zero otherwise. Here we used \( 2j + 2k = n \) in the sum.

(b) \( S = \{2, 3\} \times \{2, 3\} \times \cdots \times \{2, 3\} \) a total of \( k \) times. Since \( \Phi_{\{2,3\}}(x) = x^2 + x^3 \) we get

\[ \Phi_S(x) = (x^2 + x^3)^k \]

from the product lemma. Then the binomial theorem gives

\[ \Phi_S(x) = x^{2k} \sum_{j=0}^{k} \binom{k}{j} x^j. \]

Finally \( [x^n] \Phi_S(x) = \left( \frac{k}{n-2k} \right) \) which is valid for \( 2k \leq n \). If \( 2k > n \) then the answer is zero.

(c) \( S = [i] \times [i] \times \cdots \times [i] \). Now \( \Phi_{[i]}(x) = x + x^2 + \ldots + x^i \) which is \( x(1 - x^i)/(1 - x) \) by finite geometric series. So

\[ \Phi_S(x) = x^k(1 - x^i)^k(1 - x)^{-k}. \]

By the binomial theorem

\[ \Phi_S(x) = x^k \sum_{r=0}^{k} \sum_{s=0}^{\infty} \binom{k}{r} (-x^i)^r \binom{-k}{s} (-x)^s. \]

To get \( [x^n] \Phi_S(x) \) take \( ir + s + k = n \) i.e. take \( s = n - k - ir \). Then the answer is

\[ \sum_{r=0}^{k} \binom{k}{r} (-1)^r \binom{-k}{n-k-ir} (-1)^{n-k-ir}. \]

**Question 7.** Determine the number of compositions of \( n \) into any number of parts, where each part is odd.

**Solution.** Let \( S(k) \) be the set of sequences of positive integers of length \( k \) and \( S = \bigcup_{k=0}^{\infty} S(k) \). Then

\[ S(k) = S_1 \times S_2 \times \cdots \times S_k \]

where \( S_i = \{1, 3, 5, \ldots\} \) for all \( i \). Now

\[ \Phi_{S_i}(x) = x + x^3 + x^5 + \ldots = \frac{x}{1-x^2} \]

since it is an infinite geometric series. By the product lemma

\[ \Phi_{S(k)}(x) = \left( \frac{x}{1-x^2} \right)^k. \]
By the sum lemma
\[ \Phi_S(x) = \sum_{k=0}^{\infty} \Phi_{S(k)}(x) = \sum_{k=0}^{\infty} \left( \frac{x}{1 - x^2} \right)^k \]
since it is an infinite geometric series. Simplifying, we get
\[ \Phi_S(x) = \frac{1 - x^2}{1 - x - x^2}. \]
We can proceed by using recurrence equations or sums. By recurrence equations we get
\[ a_n = a_{n-1} + a_{n-2} \]
where \( a_1 = 1 \) and \( a_2 = 1 \). Solving this (show all working as in the notes) we get the Fibonacci numbers
\[ a_n = \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \frac{1}{\sqrt{5}} \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1}. \]
Doing it by sums, we get
\[ \Phi_S(x) = (1 - x^2) \sum_{j=0}^{\infty} (x + x^2)^j \]
since the sum of the geometric series is 1/(1 - x - x^2), and now
\[ \Phi_S(x) = (1 - x^2) \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i} x^i. \]
To get \([x^n] \Phi_S(x)\) split the sum in two:
\[ \Phi_S(x) = \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i} x^{i+j} - \sum_{j=0}^{\infty} \sum_{i=0}^{j} \binom{j}{i} x^{i+j+2}. \]
In the first let \( i = n - j \) and in the second let \( i = n - j - 2 \) to get
\[ [x^n] \Phi_S(x) = \sum_{j=0}^{n} \binom{j}{n-j} - \sum_{j=0}^{n-2} \binom{j}{n-j-2}. \]

**Question 8.** How many sets \( \{x_1, x_2, \ldots, x_k\} \subset [n] \) have \( x_{i-1} + i \leq x_i \) for \( 2 \leq i \leq k? \)

**Solution.** If we consider the sequence \((y_1, y_2, \ldots, y_{k+1}) = (x_1, x_2-x_1, \ldots, x_k-x_{k-1}, n-x_k)\) then we have a composition of \( n \) with \( k+1 \) part and where the \( i \)th part is at least \( i \) except that the last part is any non-negative integer (since \( n - x_k \geq 0 \)). Let
\[ S = S_1 \times S_2 \times \ldots \times S_{k+1} \]
where \( S_{k+1} = \mathbb{Z}^+ \) and \( S_i = \{ i, i+1, i+2, \ldots \} \) for \( i \in [k] \). Then for \( i \in [k] \),

\[
\Phi_{S_i}(x) = x^i + x^{i+1} + \ldots = \frac{x^i}{1-x}
\]

and

\[
\Phi_{S_{k+1}}(x) = 1 + x + x^2 + \ldots = \frac{1}{1-x}.
\]

Therefore by the product lemma

\[
\Phi_S(x) = \prod_{i=1}^{k} \frac{x^i}{1-x} \cdot \frac{1}{1-x} = x^{\binom{k+1}{2}}(1-x)^{-k-1}.
\]

Here we used \( x^1 \cdot x^2 \cdots x^k = x^{1+2+\ldots+k} \) and \( 1 + 2 + \ldots + k = \binom{k+1}{2} \). Now the binomial theorem gives

\[
\Phi_S(x) = x^{\binom{k+1}{2}} \sum_{j=0}^{\infty} \binom{-k-1}{j}(-x)^j.
\]

To get \([x^n]\Phi_S(x)\) put \( j = n - \binom{k+1}{2} \) so that

\[
[x^n]\Phi_S(x) = \binom{-k-1}{n - \binom{k+1}{2}}(-1)^{n-\binom{k+1}{2}}.
\]

**Question 9** Let \( S \) denote the set of unordered lists (sets with repeated elements allowed) of positive integers, with weight function \( \omega(A) = x_1 + x_2 + \ldots + x_k \). Prove that

\[
\Phi_S(x) = \prod_{r=1}^{\infty} \frac{1}{1-x^r}.
\]

**Solution.** Since

\[
\frac{1}{1-x^r} = 1 + x^r + x^{2r} + \ldots
\]

we get that

\[
\Phi_S(x) = (1 + x + x^2 + \cdots)(1 + x^2 + x^4 + \cdots)(1 + x^3 + x^6 + \cdots)(1 + x^4 + x^8 + \cdots) \cdots.
\]

Then \([x^n]\Phi_S(x)\) should be the number of sets of integers whose sum of elements is \( n \). Now each a term from the \( r \)th bracket has the form \( x^{ir} \) so

\[
n = m_1 + 2m_2 + 3m_3 + \ldots + rm_r.
\]

This means we are writing \( n \) as a sum of elements in the list \( 111111222222333333 \cdots rrrrrr \) where \( i \) appears \( m_i \) times as required.
**Question 10.** Determine the number of binary strings of length \( n \) with the given restrictions.

(a) The strings have only blocks of odd length.
(b) The strings do not contain 011.
(c) The strings do not contain 101 or 010.
(d)* The strings do not contain 101.

**Solutions.**

(a) The expression

\[
S = (\{\varepsilon\} \cup \{0\}\{00\})^*(\{1\}\{11\})^*(\{0\}\{00\})^*(\{\varepsilon\} \cup \{1\}\{11\})^*
\]

uniquely creates all the strings we want. The generating function is

\[
\Phi_S(x) = \left(1 + \frac{x}{1-x^2}\right)^2 \frac{1}{1 - \left(\frac{x}{1-x^2}\right)^2} = \frac{(1 + x - x^2)^2}{1 - 3x^2 + x^4}.
\]

By the binomial theorem

\[
\Phi_S(x) = (1 + x - x^2)^2 \sum_{j=0}^{\infty} (3x^2 - x^4)^j = (1 + x - x^2)^2 \sum_{j=0}^{\infty} \sum_{i=0}^{j} (3x^2)^i \binom{j}{i} \left(-\frac{x^2}{3}\right)^i.
\]

It is particularly unpleasant to find \( a_n = [x^n]\Phi_S(x) \) from this but it can be done. The other way is to use the recurrence

\[
a_n - 3a_{n-2} + a_{n-4} = 0.
\]

Note that \( a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 4 \) (since 111, 000, 101, 010 are the strings of length three with all blocks of odd length). The characteristic equation is \( x^4 - 3x^2 + 1 = 0 \). The roots are

\[
\sqrt{\frac{3 + \sqrt{5}}{2}} \quad \text{and} \quad -\sqrt{\frac{3 + \sqrt{5}}{2}}
\]

which means

\[
a_n = c_1 \sqrt{\frac{3 + \sqrt{5}}{2}}^n - c_2 \sqrt{\frac{3 - \sqrt{5}}{2}}^n + c_3 \sqrt{\frac{3 - \sqrt{5}}{2}}^n - c_4 \sqrt{\frac{3 + \sqrt{5}}{2}}^n.
\]

Here \( c_1, c_2, c_3, c_4 \) are constants to be determined using \( a_0, a_1, a_2, a_3 \).

(b) Generating function is

\[
\frac{1}{1 - 2x + x^3} = \sum_{j=0}^{\infty} (2x - x^3)^j = \sum_{j=0}^{\infty} \sum_{i=0}^{j} (2x)^i (-x^2/2)^i \binom{j}{i}.
\]

The coefficient of \( x^n \) is

\[
\sum_{i=0}^{n} \binom{n-2i}{i} (-1)^i 2^{n-3i}.
\]
Alternative: use recurrence equation \( a_n = 2a_{n-1} - a_{n-3} \) with initial conditions \( a_0 = 1, a_1 = 2, \) and \( a_2 = 4. \) The characteristic equation is \( \alpha^3 - 2\alpha^2 + 1 = 0 \) and \( \alpha = 1 \) is an obvious root. Then we get \((\alpha - 1)(\alpha^2 - \alpha - 1) = 0\) so \( \alpha = \frac{1}{2}(1 \pm \sqrt{5}) \). Then

\[
a_n = c_1 + c_2 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_3 \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

Finally \( c_1, c_2, c_3 \) are determined from \( a_0 = 1, a_1 = 2, \) and \( a_2 = 4. \) Thus

\[
c_1 + c_2 + c_3 = 1 \quad c_1 + c_2 \left( \frac{1 + \sqrt{5}}{2} \right) + c_3 \left( \frac{1 - \sqrt{5}}{2} \right) = 2 \quad c_1 + c_2 \left( \frac{1 + \sqrt{5}}{2} \right)^2 + c_3 \left( \frac{1 - \sqrt{5}}{2} \right)^2 = 4.
\]

Solving these equations (e.g. by linear algebra) we get \( c_1 = -1 \) and \( c_2 = (5 + 2\sqrt{5})/5 \) and \( c_3 = (5 - 2\sqrt{5})/5 \). Therefore

\[
a_n = 1 + \frac{5 + 2\sqrt{5}}{5} \left( \frac{1 + \sqrt{5}}{2} \right)^n - \frac{5 - 2\sqrt{5}}{5} \left( \frac{1 - \sqrt{5}}{2} \right)^n.
\]

(c) These strings are uniquely created by

\[
S = \{0\}^*\{11\}\{1\}^*\{00\}\{0\}^*\{1\}^*\{00\}\{0\}^*\{11\}\{1\}^*\{0\}^*.
\]

The generating function is

\[
\Phi_S(x) = \frac{1}{(1-x)^2} \cdot \frac{1}{1 - x^4/(1-x)^2} = \frac{1}{1 - 2x + x^2 - x^4} + \frac{x(1-x)^2}{1 - 2x + x^2 - x^4} = \frac{1 + x - 2x^2 + x^3}{1 - 2x + x^2 - x^4}.
\]

Opting for recurrences, we get \( a_n - 2a_{n-1} + a_{n-2} - a_{n-4} = 0 \) and \( a_0 = 1, a_1 = 2, a_2 = 4, \) and \( a_3 = 6. \) The characteristic equation has four distinct roots: \((1 \pm \sqrt{5})/2\) and \((1 \pm \sqrt{3}i)/2\). So

\[
a_n = c_1 \left( \frac{1 + \sqrt{5}}{2} \right)^n + c_2 \left( \frac{1 - \sqrt{5}}{2} \right)^n + c_3 \left( \frac{1 + \sqrt{3}i}{2} \right)^n + c_4 \left( \frac{1 - \sqrt{3}i}{2} \right)^n.
\]

Use the initial conditions now to find \( c_1, c_2, c_3, c_4. \)