Question 1.

(a) \( \delta(G) = \kappa(G) = \lambda(G) = 2, \kappa(u, v) = 2 = \lambda(u, v). \)

(b) \( \delta(G) = 5, \kappa(G) = 2, \lambda(G) = 4, \kappa(u, v) = 2, \lambda(u, v) = 4. \)

Question 4. If the graph is complete, then it is \( K_n \) for some \( n \geq 4 \), which clearly contains a cycle of length four. If the graph is not complete, pick two nonadjacent vertices \( u, v \).

By Menger’s Theorem, there are at least three pairwise internally disjoint \( uv \) paths say \( P_1, P_2, P_3 \). Each pair of paths makes a cycle, so we get three cycles \( C_1, C_2, C_3 \). Then

\[
|C_1| + |C_2| + |C_3| = 2|P_1| + 2|P_2| + 2|P_3|.
\]

Since the right hand side is an even number, one of the cycles \( C_i \) must have even length, as required.

Question 6. By Menger’s Theorem, if \( \kappa(G) = \min \kappa(u, v) \) and \( \lambda(G) = \min \lambda(u, v) \). Since \( \kappa(u, v) \leq \lambda(u, v) \), we know that \( \kappa(G) \leq \lambda(G) \). However, any two edge-disjoint \( uv \) paths are actually internally disjoint, for if they share an internal vertex then that vertex would have to have degree at least four, contradicting that \( G \) has maximum degree three. Therefore \( \kappa(u, v) = \lambda(u, v) \) and so \( \kappa(G) = \lambda(G) \).

Question 10. All working is required at each stage when building a path from source to sink and increasing the flow. State a maximum flow by listing flow in all arcs. The max flow is 3 in this case, so a minimum cut has capacity three. An example of a minimum cut is to cut off the top right two vertices of the network (the sink and the vertex from which the capacity to the sink is four).

Question 11. Show all working to find paths. State the flow precisely. A max flow has value two and a cut is \( S = \{s\} \) with capacity two.