Solution to Additional Problem 1(b)  

We want to find all extremes of \( f(x, y, z) = x^3 + y^3 + z^3 - 3xyz \). First we determine the critical points of \( f \): since \( \nabla f = (3x^2 - 3yz, 3y^2 - 3xz, 3z^2 - 3xy) \), critical points occur when

\[
\begin{align*}
3x^2 &= 3yz & (1) \\
3y^2 &= 3xz & (2) \\
3z^2 &= 3xy. & (3)
\end{align*}
\]

Multiplying (1) by \( x \) and (2) by \( y \) and subtracting the results we get \( 3(x^3 - y^3) = 3xyz - 3xyz = 0 \) so \( y = x \). Similarly, we get \( y = z \) using (2) and (3) so \( x = y = z \). Therefore the critical points are all points of the form \( (x, x, x) \).

We use the second derivative test to determine the nature of the critical points. We have

\[
\begin{align*}
f_{xx} &= 6x & f_{xy} &= -3z & f_{xz} &= -3y \\
f_{yy} &= 6y & f_{yz} &= -3x & f_{zz} &= 6z.
\end{align*}
\]

At the points \( (x, x, x) \) this gives the Hessian matrix

\[
\begin{pmatrix}
6x & -3x & -3x \\
-3x & 6x & -3x \\
-3x & -3x & 6x
\end{pmatrix}
\]

The nature of the critical points is determined by the signs of the determinants of the matrices

\[
H_3 = \begin{pmatrix}
6x & -3x & -3x \\
-3x & 6x & -3x \\
-3x & -3x & 6x
\end{pmatrix} \quad H_2 = \begin{pmatrix}
6x & -3x \\
-3x & 6x
\end{pmatrix} \quad H_1 = (6x).
\]

We note that \( \det(H_1) = 6x \), \( \det(H_2) = 27x^2 \), \( \det(H_3) = 162x^3 \). If \( x > 0 \), then these are all positive so \( (x, x, x) \) represents a minimum for \( f \). If \( x < 0 \), then these are alternately negative and positive, which means \( (x, x, x) \) represents a maximum for \( f \). If \( x = 0 \), then the second derivative test fails. Now observe \( f(0, 0, 0) = 0 \), whereas \( f(x, 0, 0) = x^3 \) can be negative or positive near \( (0, 0, 0) \). This means \( (0, 0, 0) \) is neither a minimum nor a maximum of \( f \).
Solution to Additional Problem 2(e)

We want to find all extreme points of \( f(x, y) = e^{-x^2} \sin(x^2 + y^2) \). First we find all critical points. Note that
\[
\nabla f = (-2xe^{-x^2} \sin(x^2 + y^2) + 2xe^{-x^2} \cos(x^2 + y^2), 2ye^{-x^2} \cos(x^2 + y^2)).
\]
This is zero when
\[
\begin{align*}
 x &= 0 \quad \text{or} \quad \sin(x^2 + y^2) = \cos(x^2 + y^2) \quad \text{and} \\
 y &= 0 \quad \text{or} \quad \cos(x^2 + y^2) = 0
\end{align*}
\]
where \( k \in \mathbb{Z} \). Therefore
\[
\begin{align*}
 x &= 0 \quad \text{or} \quad x^2 + y^2 = \frac{\pi}{4} + k\pi \quad \text{and} \\
 y &= 0 \quad \text{or} \quad x^2 + y^2 = \frac{\pi}{2} + k\pi
\end{align*}
\]
So the critical points are
\[
(0, 0) \quad (0, \sqrt{\frac{\pi}{2} + k\pi}) \quad (\sqrt{\frac{\pi}{4} + k\pi, 0})
\]
where we insist \( k \in \mathbb{Z} \) and \( k \geq 0 \). Next we determine the nature of these critical points. We have
\[
\begin{align*}
f_{xx} &= (2 - 4x^2)e^{-x^2}(\sin(x^2 + y^2) - \cos(x^2 + y^2)) - 4x^2e^{-x^2}(\sin(x^2 + y^2) + \cos(x^2 + y^2)) \\
f_{yy} &= 2e^{-x^2}\cos(x^2 + y^2) - 2ye^{-x^2}\sin(x^2 + y^2) \\
f_{xy} &= -4xye^{-x^2}\cos(x^2 + y^2) - 4xye^{-x^2}\sin(x^2 + y^2).
\end{align*}
\]
We immediately observe \( f_{xy} = 0 \) at all critical points. At \((0, 0)\) we have \( f_{xx} = -2 \) and \( f_{yy} = 2 \). Therefore the Hessian matrix is
\[
H = \begin{pmatrix}
-2 & 0 \\
0 & 2
\end{pmatrix}.
\]
Since \( H \) is negative definite, this means \((0, 0)\) represents a maximum of \( f \), and \( f(0, 0) = 0 \). Next we check the critical points \((0, \sqrt{\pi/2 + k\pi})\). If \( k \) is even, we get \( \sin(x^2 + y^2) = \sin(\pi/2 + k\pi) = 1 \) at this critical point. Therefore for \( k \) even:
\[
H = \begin{pmatrix}
2 & 0 \\
0 & -2\sqrt{\pi/2 + k\pi}
\end{pmatrix}.
\]
This matrix is negative definite so \((0, \sqrt{\pi/2 + k\pi})\) represents a local maximum of \(f\), namely \(f(0, \sqrt{\pi/2 + k\pi}) = 1\). If \(k\) is odd, we get \(\sin(x^2 + y^2) = \sin(\pi/2 + k\pi) = -1\) at this critical point. Therefore for \(k\) odd,

\[
H = \begin{pmatrix}
2 & 0 \\
0 & 2\sqrt{\pi/2 + k\pi}
\end{pmatrix}.
\]

So this is a positive definite matrix, and therefore \((0, \sqrt{\pi/2 + k\pi})\) represents a local minimum of \(f\), namely \(f(0, \sqrt{\pi/2 + k\pi}) = -1\). Moving on to the critical points \((\sqrt{\pi/4 + k\pi}, 0)\), we observe that if \(k\) is even then \(\sin(x^2 + y^2) = \cos(x^2 + y^2) = 1/\sqrt{2}\) at this critical point. Therefore, if \(k\) is even,

\[
H = \begin{pmatrix}
-4\sqrt{2}(\pi/4 + k\pi)e^{-\pi/4-k\pi} & 0 \\
0 & \sqrt{2}e^{-\pi/4-k\pi}
\end{pmatrix}.
\]

This matrix is negative definite, so \(f(\sqrt{\pi/4 + k\pi}, 0) = \frac{1}{\sqrt{2}}e^{-\pi/4-k\pi}\) is a local maximum if \(k\) is even. We observe that if \(k\) is odd then \(\sin(x^2 + y^2) = -\cos(x^2 + y^2) = -1/\sqrt{2}\) at this critical point. Therefore, if \(k\) is odd,

\[
H = \begin{pmatrix}
-\sqrt{2}(2 - 4(\pi/4 + k\pi))e^{-\pi/4-k\pi} & 0 \\
0 & \sqrt{2}e^{-\pi/4-k\pi}
\end{pmatrix}.
\]

This matrix is positive definite, so \(f(\sqrt{\pi/4 + k\pi}, 0) = -\frac{1}{\sqrt{2}}e^{-\pi/4-k\pi}\) represents a local minimum if \(k\) is odd. If we want the absolute maximum and minimum of \(f\), then the absolute minimum of \(-1\) is at \((0, \sqrt{\pi/2 + k\pi})\) for \(k\) even, and the absolute maximum is \(\frac{1}{\sqrt{2}}e^{-\pi/4-k\pi}\) which occurs at \((\sqrt{\pi/4 + k\pi}, 0)\). A picture is shown below: