Q1	Q2	Q3	Q4	Q5	Q6	Q7	Q8	Total
6	4	5	6	7	7	9	6	50

Final Examination

Math 20E – Vector Calculus

Instructor – J. Verstraete

Allotted time – 3 hours

Answers are to be written clearly and legibly
Calculators are allowed
State clearly any theorems used without proof
Total 50 points

Student Name

SOLUTIONS.

Student ID Number

6

Question 1.

(a) Define $\lim_{x \to a} f(x) = L$ when $f : \mathbb{R}^n \to \mathbb{R}$. $\boxed{2}$

(b) Prove that the following limit does *not* exist: $\boxed{4}$

$$\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{x^2+y^2}.$$

(a) Bookwork 0 or 2 points

(b) Put $y = mx$. Then

$$\frac{(x-y)^2}{(x^2+y^2)} = \frac{(1-m)^2 x^2}{(1+m^2)x^2} = \frac{(1-m)^2}{1+m^2}$$

So $\lim_{x \to 0} \frac{(1-m)^2}{1+m^2} = \frac{(1-m)^2}{1+m^2}$ depends on m

Therefore original limit does not exist.

Question 2.

Find the *maximum value* of the function $f(x, y) = x + y$ subject to the constraint $x^2 + y^2 = 2$.

$\boxed{4}$

Use Lagrange Multipliers.

$\phi(x, y, \lambda) = x + y + \lambda(2 - x^2 - y^2)$ ✓½

Extremes occur at $\nabla\phi = 0$ ½ so

$\frac{\partial\phi}{\partial x} = 0 \rightarrow 1 - 2x\lambda = 0$ ✓½

$\frac{\partial\phi}{\partial y} = 0 \Rightarrow 1 - 2y\lambda = 0$ ✓½

Since $\lambda \neq 0$, we get $x = y = \frac{1}{2\lambda}$ ✓½ ✓½

Since $x^2 + y^2 = 2$ we get $\lambda = \frac{1}{2}$, $x = y = 1$ ✓½

So $x = y = 1$ is an extreme. It is a maximum since $f(0, \sqrt{2}) = \sqrt{2} < f(1,1) = 2$.

Maximum value is $f(1,1) = 2$ ✓½

$\boxed{5}$

Question 3.

(a) State *Fubini's Theorem* for a continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ on a rectangle $[a, b] \times [c, d]$. $\boxed{2}$

(b) Evaluate the double integral

$$\int_0^{2\pi} \int_0^{2\pi} \sin(x + \sin y) dy dx. \quad \boxed{3}$$

(a) let f be a continuous $\frac{1}{2}$ function on $[a,b] \times [c,d]$.

Then $\int_a^b \int_c^d f(x,y)\, d\cancel{x}\, d\cancel{y} = \int_c^d \int_a^b f(x,y)\, dy\, dx$ $\frac{1}{2}$ $\frac{1}{2}$

$= \iint_{[a,b]\times[c,d]} f(x,y)\, dA$ $\frac{1}{2}$ 2

(b) By Fubini's Theorem, since $\sin(x+\sin y)$ is continuous on $[0,2\pi] \times [0,2\pi]$,

$$\int_0^{2\pi} \int_0^{2\pi} \sin(x+\sin y)\, dy\, dx$$

$$= \int_0^{2\pi} \int_0^{2\pi} \sin(x+\sin y)\, dx\, dy \qquad 3$$

$$= \int_0^{2\pi} -\cos(x+\sin y)\Big]_0^{2\pi} dy$$

$$= \int_0^{2\pi} 0\, dy = 0.$$

$\boxed{6}$

Question 4.

Consider the transformation from Cartesian coordinates (x, y) to coordinates (u, v) given by the following equations:

$$u = xy \quad v = y^{1/3} - x^{1/3}.$$

(a) Find the *Jacobian determinant* for this transformation. $\boxed{4}$

(b) For what values of x and y is the transformation invertible? $\boxed{2}$

(a) Jacobian determinant is

$$\frac{1}{\left|\begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{pmatrix}\right|} \checkmark = \frac{1}{\left|\begin{pmatrix} y & x \\ -\frac{1}{3}x^{-2/3} & \frac{1}{3}y^{-2/3} \end{pmatrix}\right|}$$

$$= \frac{3}{y^{1/3} + x^{1/3}} \checkmark\checkmark$$

4

(b) Jacobian must be defined and non-zero.

So okay if $x^{1/3} \neq -y^{1/3}$

ie $x \neq -y$

So All (x, y) with $x \neq -y$. $\checkmark\checkmark$

2

7

Question 5.

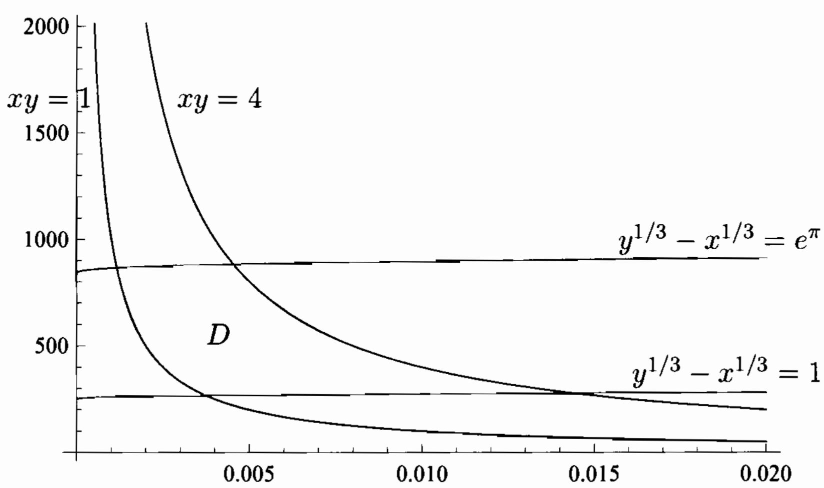
Use the transformation in Question 4 to evaluate the integral

$$\iint_D \sin(\ln(y^{1/3} - x^{1/3}))dA$$

where

$$D = \{(x, y) : 1 \leq xy \leq 4, 1 \leq y^{1/3} - x^{1/3} \leq e^{\pi}, x \geq 0, y \geq 0\}.$$

The curves bounding the region D are shown below.



Change of variables theorem

integral $= \int_1^4 \int_1^{e^\pi} \sin(\ln v) \frac{1}{v} \cdot 3 \, dv \, du$

since $D = \{(u,v) : 1 \leq u \leq 4, 1 \leq v \leq e^\pi\}$.

Let $z = \ln v$,

integral $= \int_1^4 \int_0^\pi 3 \sin z \, dz \, du$

$= \int_1^4 3 \cdot 2 \, du = 3 \cdot 6 = 18$.

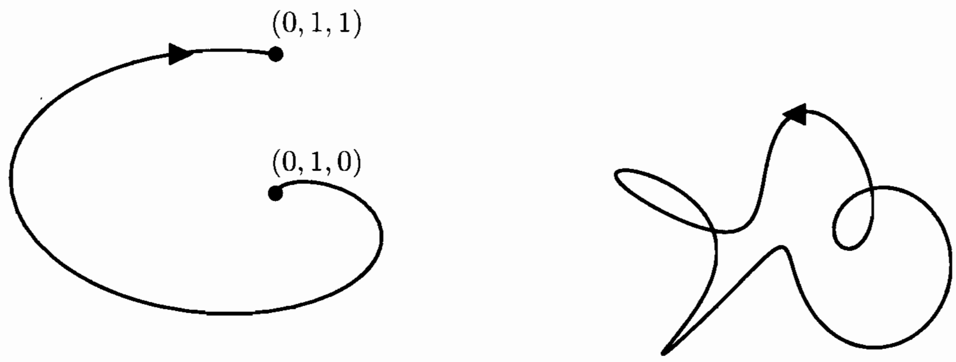
Question 5 continued...

Question 6.

$\boxed{6}$

(a) Prove that the vector field $\mathbf{f}(x, y, z) = (y, x, z)$ is *conservative.* $\boxed{2}$

(b) Evaluate the line integral $\int_\gamma \mathbf{f} \cdot d\mathbf{r}$ for each of the *oriented curves* γ below: $\boxed{4}$



(a) $\nabla \times f = \vec{0}$ since

$$\nabla \times f = \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y & x & z \end{vmatrix}$$

$$= i\left(\underbrace{\frac{\partial z}{\partial y}}_{0} - \underbrace{\frac{\partial y}{\partial z}}_{0}\right) - j\left(\underbrace{\frac{\partial z}{\partial x}}_{0} - \underbrace{\frac{\partial y}{\partial z}}_{0}\right) + k\left(\underbrace{\frac{\partial x}{\partial x}}_{1} - \underbrace{\frac{\partial y}{\partial y}}_{1}\right)$$

$$= (0,0,0)$$ ✓ 2

$\boxed{0.5}$ Find potential F:

$F_x = y \to F = xy + c$ ✓ $\frac{1}{2}$

$F_y = x \to F = xy + d$ ✓ $\frac{1}{2}$

$F_z = z \to F = \frac{z^2}{2} + e$ ✓ $\frac{1}{2}$

let $c = d = \frac{z^2}{2}$, $e = xy$ so

$F = xy + \frac{z^2}{2}$ ✓ $\frac{1}{2}$ 2.

(b) Second line integral is 0 since f is conservative and γ is closed. ✓

Use fundamental theorem for first 4

$$\int_\gamma f \cdot dr = F(0,1,1) \quad F(0,1,0)$$

$$= \frac{1}{2}$$ ✓

Question 7.

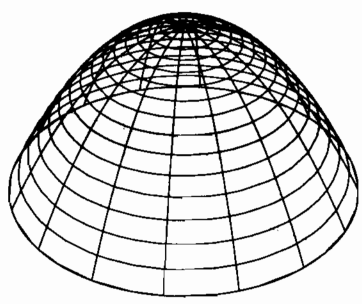
$\boxed{9}$

(a) State *Stokes' Theorem.* $\boxed{3/2}$

(b) Let $\mathbf{f}(x, y, z) = (x^2 + y^2 + z^2, x + y + z, z)$ and let Σ be the surface $z = 1 - x^2 - y^2$ for $z \geq 0$ with upward orientation, shown below. Evaluate the surface integral $\boxed{15/2}$

$$\iint_{\Sigma} \nabla \times \mathbf{f} \cdot d\mathbf{R}$$

You may use the fact that $\displaystyle\int_0^{2\pi} \cos^2 t dt = \pi$.



(a) $3/2$ $\quad \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$

(b) $\iint_{\Sigma} (\nabla \times f) \cdot dR = \int_{\gamma} f \cdot dP$

where γ is a counterclockwise $\frac{1}{2}$ circle in the xy plane. so

$\gamma: r(t) = (\cos t, \sin t, 0)$ is the $0 \leq t \leq 2\pi$

parametrization

On γ, $f = (1, \cos t + \sin t, 0)$ so

$$\int_{\gamma} f \cdot dr = \int_0^{2\pi} (1, \cos t + \sin t, 0) \cdot (-\sin t, \cos t, 0)\, dt$$

$$= \int_0^{2\pi} \cos^2 t \, dt$$

$$= \pi$$

Question 7 continued...

$\boxed{6}$

Question 8.

(a) State the *divergence theorem.* $\boxed{3}$

(b) Let Σ denote the surface of the unit box $[0,1] \times [0,1] \times [0,1]$, with outward orientation. Determine

$$\iint_\Sigma \mathbf{f} \cdot d\mathbf{R}$$

$\boxed{3}$

where

$$\mathbf{f}(x, y, z) = (x^2 + e^z, y^2 + e^z, z^2).$$

(a) 3

(b) $\iint_\Sigma f \cdot dR = \iiint_D (\nabla \cdot f)\, dV$ ✓

$= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\, dz\, dy\, dx$ ✓

$= 3$ ✓ 3

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